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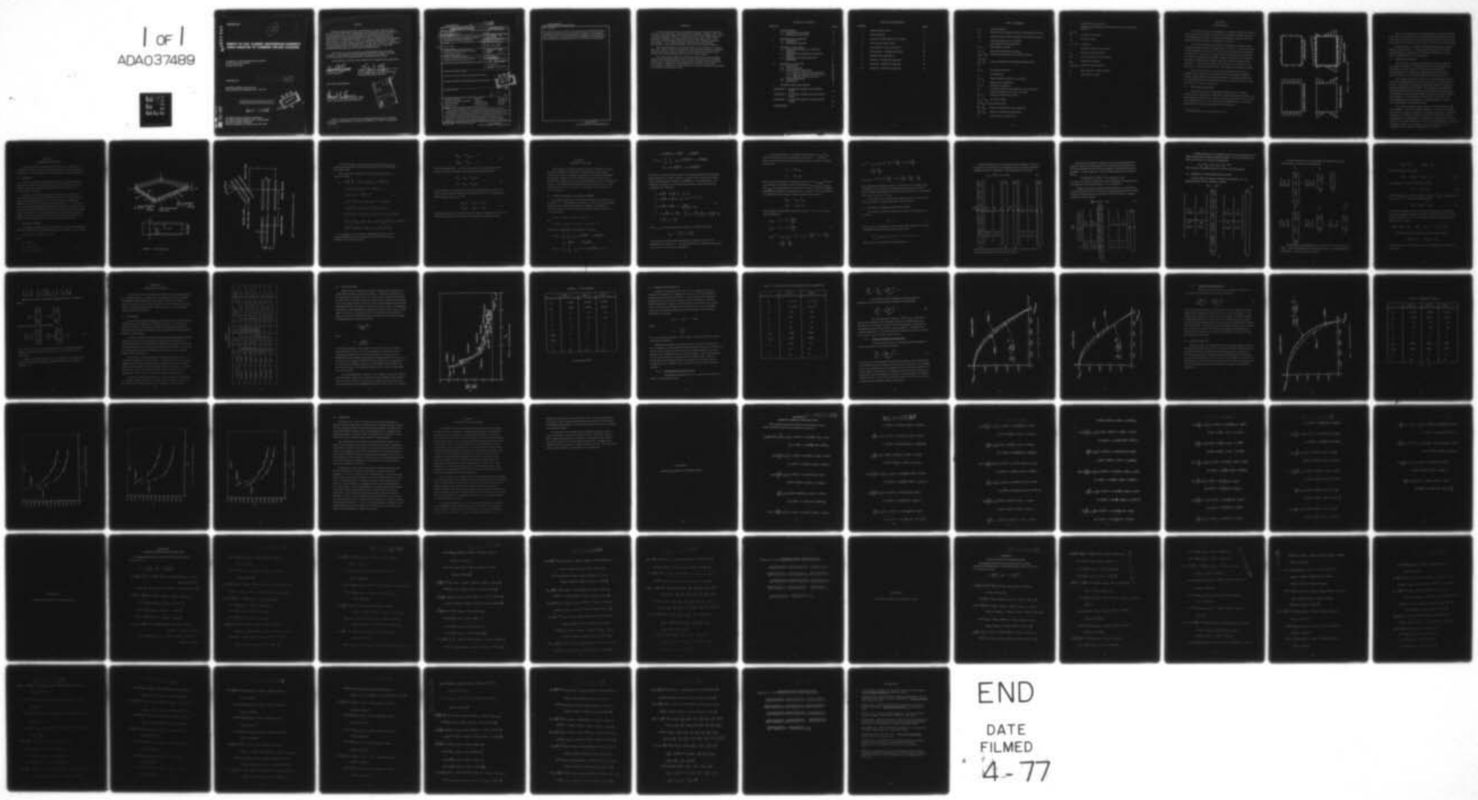

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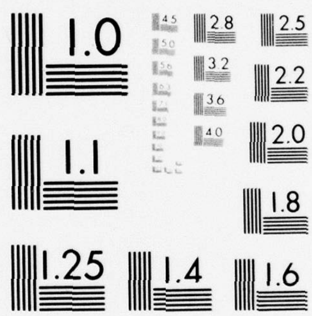

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STABILITY OF FLAT, CLAMPED, RECTANGULAR SANDWICH PANELS SUBJECTED TO COMBINED INPLANE LOADINGS

UNIVERSITY OF DAYTON RESEARCH INSTITUTE
300 COLLEGE PARK AVENUE
DAYTON, OHIO 45469

DECEMBER 1976

TECHNICAL REPORT AFFDL-TR-76-137
FINAL REPORT FOR PERIOD MAY 1975 - MAY 1976

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AIR FORCE FLIGHT DYNAMICS LABORATORY
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The general stability of flat, rectangular sandwich panels loaded by combined biaxial compression, biaxial edgewise bending and edgewise shear is investigated. The analysis is applicable to sandwich structures having isotropic face sheets and orthotropic cores. The analysis procedure is based upon a linear potential energy formulation and the Ritz method of discretization. For the case of clamped boundaries, it is shown that the equation of the potential energy must be minimized by the use of matrices.

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The analysis leads to an infinite series solution for which the eigenvalues of the equation of matrices are monotonically decreasing. This analysis is presented in detail with the equations for the potential energy presented in three Appendices. The equation presented in Appendix C is the expression used to minimize the potential energy of a flat, clamped sandwich panel.

PREFACE

The work reported herein was performed by the Aerospace Mechanics Division of the University of Dayton Research Institute in Dayton, Ohio, under Air Force Contract F33615-75-C-3009, for the Air Force Flight Dynamics Laboratory (AFFDL) at Wright-Patterson Air Force Base, Ohio. This effort was conducted under Task 02 of Project 1368, "Structural Sandwich Composites." The technical direction and support was provided by Mr. Harold C. Croop (AFFDL/FBS) as the Air Force Project Engineer.

The work described was conducted during the period May 1975 through May 1976, under the general supervision of Mr. Dale H. Whitford, Aerospace Mechanics Division, and Mr. George J. Roth, Structural Analysis Group Leader. The principal investigator was Dr. Fred K. Bogner.

The author acknowledges the University of Dayton Research Institute Library for providing reference material, and the secretaries and Graphic Arts Section of the University for their expert assistance in the preparation of this report. Thanks are also extended to Dr. Fred K. Bogner and Mr. Robert A. Brockman for their efforts in reviewing the material presented for technical accuracy and clarity.

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LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

a, b	panel dimensions
a_{ij}	elements of the coefficient matrix in the potential energy
b_{ij}	elements of the coefficient matrix in the work expression
c	subscript denoting core quantities
d	total depth of the sandwich panel
e_x, e_y	neutral plane locations
$f = 1, 2$	subscript denoting face quantities
H_{mn}, K_{mn}	undetermined parameters in potential energy
n_{xo}, n_{xb}, n_{xy}, n_{yb}, n_{yo}	relative magnitudes of the applied inplane loads
t_1, t_2	face sheet thicknesses
t_c	core thickness
u_n, v_n	inplane displacements due to stretching
u, v, w	displacement components
x, y, z	Cartesian coordinates
D	coefficient matrix in the potential energy expression
E	coefficient matrix in the work expression
E_1, E_2	Young's moduli of face sheets
G_{cxz}, G_{cyz}	core shear moduli
$\bar{N}_x$, $\bar{N}_{xy}$, $\bar{N}_y$	total applied loads
$\bar{N}_{xo}$, $\bar{N}_{yo}$	edgewise compression load components
$\bar{N}_{xb}$, $\bar{N}_{yb}$	edgewise bending load components
W	work done by external forces

A	assumed-mode parameters
,	indicates partial differentiation with respect to the parameter following
α_{fx}, α_{fy}, β_f	geometric parameters
δ_{ij}	Kronecker delta
$\lambda_f = 1 - \nu_f^2$	eigenvalue
ν_f	Poisson's ratios for face sheets
π_p	potential energy functional
ϕ, ψ	core rotations in xz and yz planes
Δ_{ij}	defined delta function
B_{mn}, C_{mn}	assumed-mode parameters
G	final geometric stiffness matrix
K	final stiffness matrix

SECTION 1

INTRODUCTION

One of the primary considerations in the design of structural sandwich components is the stability of the panel when subjected to inplane loadings. These applied loads consist of edgewise compression, edgewise shear and edgewise bending moments, as illustrated in Figure 1.

The amount of literature available concerning the buckling of flat, retangular sandwich panels is rather limited. There are several references to uniaxial edgewise compression[1*] and several references to the problem of shear alone[2]. The case of combined edgewise bending and compression has not been extensively investigated. Other observations have considered the effects of compression combined with shear by the use of interaction formulas[6]. The results presented, therefore, have very little in the way of previous results that would in some way support the material presented.

This report presents a method for the prediction of the general stability of flat, clamped sandwich panels loaded by combined, edgewise bending, biaxial compression, edgewise shear, and uniaxial compression. It is anticipated that the method presented will be a useful tool in the design and analysis of sandwich components.

1.1 DEFINITION AND SCOPE

The analysis presented considers buckling of flat, rectangular sandwich panels subjected to combinations of uniaxial edgewise compression, biaxial edgewise compression, edgewise bending, and edgewise shear loading. The component layers are restricted to linear elastic behavior and small displacements for the clamped panels being considered.

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* Numerical superscripts indicate References.

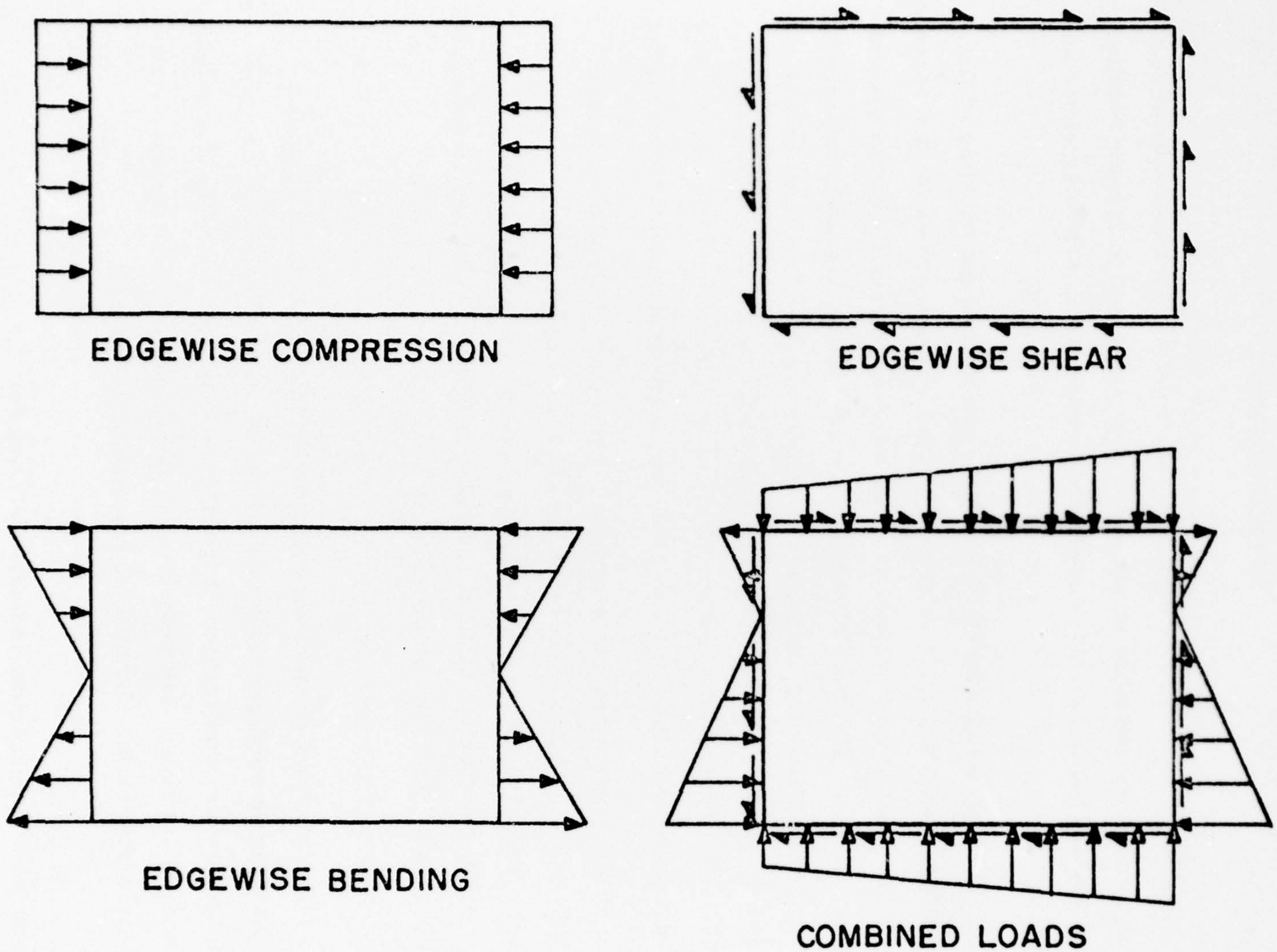

Figure 1. Applied Inplane Loads.

The core is considered to be completely rigid in the direction normal to the panel. The resistance of the core to extension and bending is assumed to be negligible; therefore, the strain energy may be taken to consist only of contributions due to transverse shear moduli of the core which may be helpful.

The face sheets are assumed to be isotropic but may have different properties and thicknesses. Critical combined loads of classical plates may be obtained as a special case (Love-Kirchoff hypothesis), since the flexural rigidity of the faces is not neglected.

1.2 METHOD OF SOLUTION

Linear analyses of sandwich deformations, in general, fall into two categories as follows:[4] formulations in terms of face displacement, and analyses in terms of functions related to core displacements. Theoretically, the two methods are equivalent, but with regard to the boundary conditions, permit slightly different numerical treatments.

The results presented are based on the so-called "tilting method"[3], by which the displacements in each face are related to the displacement of the core. The solution is obtained by using the Ritz method to minimize the total potential energy in the panel. For the purpose of uncoupling the flexural and extensional strain energies, it is assumed that there exists a "neutral plane" in which no inplane displacements occur during the buckling mode deflection. The neutral plane location which corresponds to the given buckling mode is then treated as an additional degree of freedom in the minimization of the potential energy.

It is clearly noted that a linear analysis is only valid under certain ideal conditions. The most important conditions are that prior to buckling, the structural components experience only infinitesimal changes in geometry, and that there is sufficient symmetry such that the assumptions aimed at uncoupling modes of deformation are justified (i.e., the existence of neutral planes). Also sandwich panels which have severely unbalanced face construction are likely to require a more rigorous, nonlinear treatment, since the assumptions concerning the neutral planes may be exceeded.

SECTION 2
THEORETICAL APPROACH

To construct the total potential energy of the panel, a displacement formulation is used. The theory on which this analysis is based is presented in Reference 4. For completeness the essential equations are presented here.

2.1 GEOMETRY

In Figure 2, the panel configuration, reference axes and loading conventions are indicated. The x-y plane is chosen to coincide with the panel's midplane. The stress resultants and displacements are shown to be of positive sense in the figure.

In Figure 3, the assumed geometry of deformation is shown in the xz coordinate planes. A transverse deflection occurs, accompanied by transverse shear deformations in the core as the panel buckles. During this deflection it is assumed that there is some plane located at $z = e_x$ where there is no further inplane displacement and that the rotations of the panel in the xz plane occur about points in this plane whereas rotations occur around the plane $z = e_y$ in the yz plane. With these assumptions, each of the inplane displacements of the plane can be expressed in terms of the transverse deflection w and the core rotations ϕ and ψ in the xz and yz planes respectively.

2.2 POTENTIAL ENERGY

The assumed geometry (Figure 3) permits the rotations and displacement about the neutral plane to have the following forms (see Reference 4 for details):

$$\phi = \phi(x, y), \quad \psi = \psi(x, y), \quad w = w(x, y)$$

and

$$\begin{aligned} \beta_f &= -\frac{d}{2} + t_f \\ \alpha_{fx} &= \beta_f + (-1)^f e_x \\ \alpha_{fy} &= \beta_f + (-1)^f e_y ; \; f = 1, 2 \end{aligned} \tag{1}$$

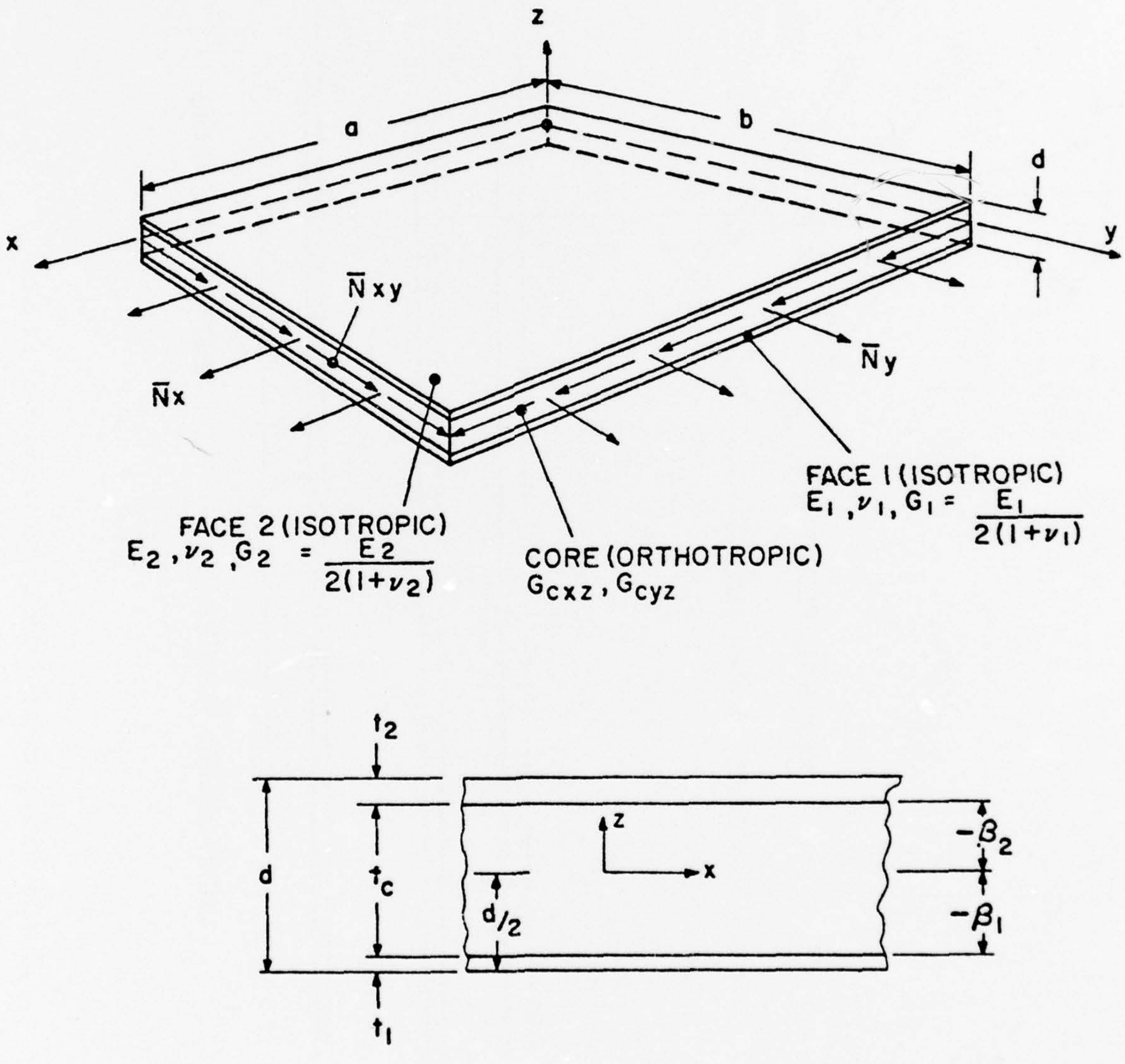

Figure 2. Panel Geometry.

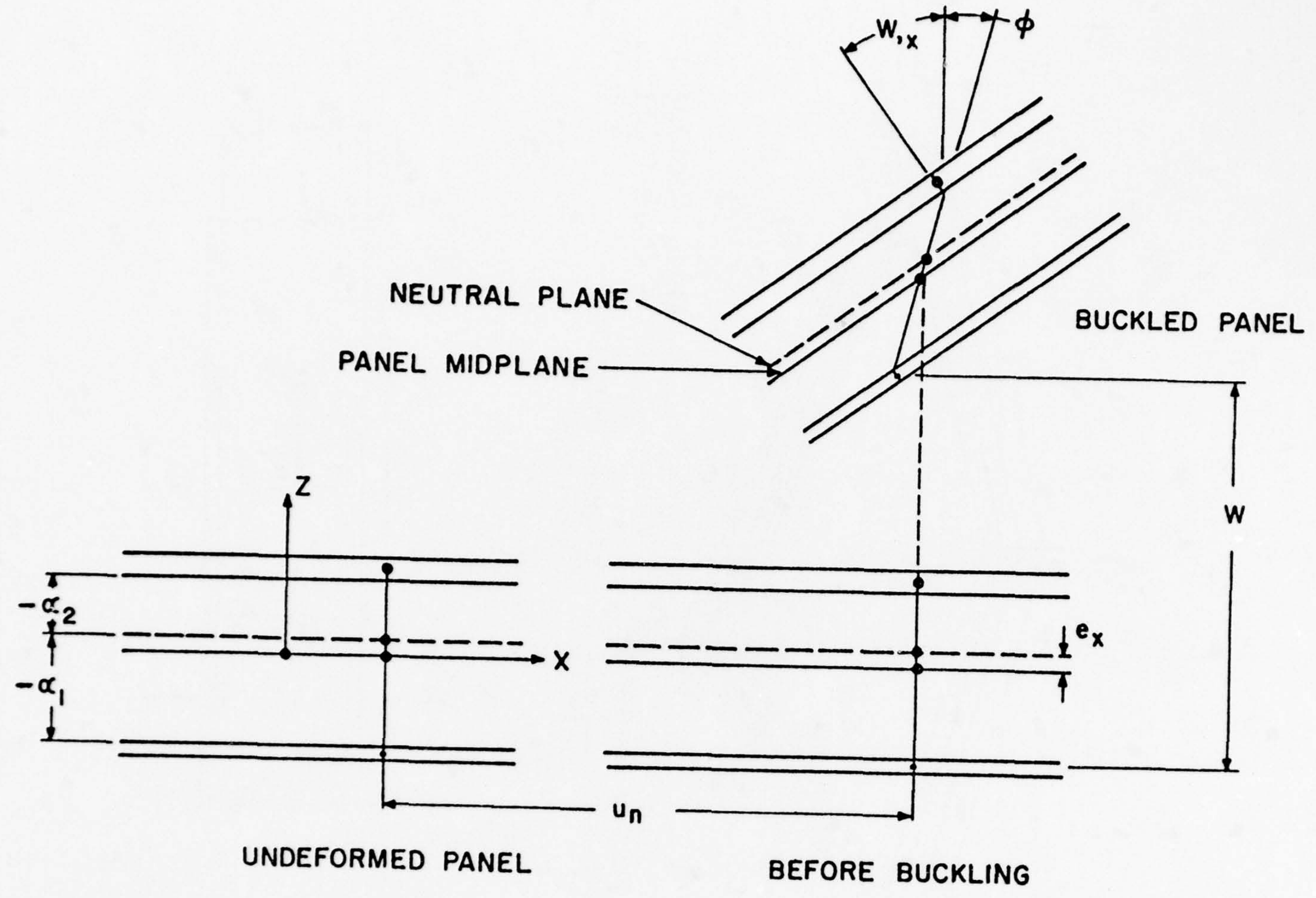

Figure 3. Geometry of Deformation in x-z Plane.

It is again important to note that the neutral plane locations e_x, e_y are unknown parameters which will be determined in the process of minimizing the potential energy.

Thus making these substitutions,[4] the potential energy functional is obtained as follows:

$$
\begin{aligned}
\pi_p = \frac{1}{2} \sum_{f=1}^{2} \frac{E_f}{\lambda_f} & \int_o^b \int_o^a \{ \alpha_{fx}^2 t_f (\phi,_x{}^2 + \frac{1-\nu_f}{2} \phi,_y{}^2) \\
& + \alpha_{fx} \alpha_{fy} t_f [2 \nu_f \phi,_x \psi,_y + (1 - \nu_f) \phi,_y \psi,_x] \\
& + \alpha_{fy}^2 t_f (\psi,_y{}^2 + \frac{1 - \nu_f}{2} \psi,_x{}^2) \qquad (2) \\
& + \alpha_{fx} t_f^2 [\phi,_x w,_{xx} + \nu_f \phi,_x w,_{yy} + (1 - \nu_f) \phi,_y w,_{xy}] \\
& + \alpha_{fy} t_f^2 [\psi,_y w,_{yy} + \nu_f \psi,_y w,_{xx} + (1 - \nu_f) \psi,_x w,_{xy}] \\
& + \frac{1}{3} t_f^3 [w,_{xx}{}^2 + 2\nu_f w,_{xx} w,_{yy} + w,_{yy}{}^2 + 2 (1 - \nu_f) w,_{xy}{}^2] \} dxdy \\
& + \frac{1}{2} \int_o^b \int_o^a [G_{cxz} t_c (\phi^2 + 2\phi w,_x + w,_x{}^2) + G_{cyz} t_c (\psi^2 + 2\psi w,_y + w,_y{}^2)] dxdy \\
& + \frac{1}{2} \int_o^b \int_o^a [\bar{N}_x w,_x{}^2 + 2 \bar{N}_{xy} w,_x w,_y + \bar{N}_y w,_y{}^2] \, dxdy
\end{aligned}
$$

It is important to realize that the resultants $\bar{N}_x$, $\bar{N}_{xy}$, $\bar{N}_y$ need not be constants throughout the panel, although they must satisfy the equilibrium conditions before buckling as follows:

$$\bar{N}_{x,x} + \bar{N}_{xy,y} = 0$$
$$\bar{N}_{xy,x} + \bar{N}_{y,y} = 0 \qquad (3)$$

Hence the quantities $\bar{N}_x$ and $\bar{N}_y$ may be taken to be both uniform and linearly varying compressive forces; having the following forms:

$$\bar{N}_x = \bar{N}_{xo} + \bar{N}_{xB}(y)$$
$$\bar{N}_y = \bar{N}_{yo} + \bar{N}_{yB}(x) \qquad (4)$$

thus permitting the consideration of edgewise bending forces. The special case of a pure edgewise bending moment (Figure 1) will be indicated by the symbols $\bar{N}_{xB}$ and $\bar{N}_{yB}$, such that

$$\bar{N}_{xB}(y) = \bar{N}_{xB}\left(1 - 2\frac{y}{b}\right)$$
$$\bar{N}_{yB}(x) = \bar{N}_{yB}\left(1 - 2\frac{x}{a}\right) \qquad (5)$$

The linearly varying component of the pure edgewise bending load, therefore, is twice the magnitude of the uniform component, and opposite in sign.

SECTION 3
NUMERICAL SOLUTION

The principle of minimum potential energy can be applied to produce the desired conditions for equilibrium. The potential energy should be expressed in terms of a complete, admissible set of functions that satisfy the geometry. Application of the principle of minimum potential energy produces the appropriate minimizing relationships in the form of an eigenvalue problem. By considering successively larger solution sets, each of which contains the previous one, monotonic convergence to the true solution is surely guaranteed.

3.1 POTENTIAL ENERGY BY THE RITZ METHOD

The potential energy associated with the buckling of the panel is given by Equation 2. This Equation is in terms of w, ϕ, and ψ, thus admissible assumed mode functions must satisfy the conditions of continuity and differentiability, as well as the following imposed boundary conditions[4] for the clamped panel:

$$w(o,y) = w(a,y) = w(x,o) = w(x,b) = 0$$

and

$$w,_x(o,y) = w,_x(a,y) = w,_y(x,o) = w,_y(x,b) = 0.$$

Therefore, an appropriate set of functions is as follows:

$$w(x,y) = \sum_{m=1}^{\ell} \sum_{n=1}^{\ell} A_{mn} \left(\cos\frac{(m-1)\pi x}{a} - \cos\frac{(m+1)\pi x}{a}\right) \left(\cos\frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right) \tag{6}$$

$$\phi(x,y) = \sum_{m=1}^{\ell} \sum_{n=1}^{\ell} B_{mn} \Big((m+1)\sin\frac{(m+1)\pi x}{a} - (m-1)$$

$$\sin \frac{(m-1)\pi x}{a}\Big)\left(\cos \frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right)$$

$$\psi(x,y) = \sum_{m=1}^{\ell} \sum_{n=1}^{\ell} C_{mn} \left(\cos \frac{(m-1)\pi x}{a}\right) - \cos \frac{(m+1)\pi x}{a}\Big)$$

$$\left((n+1) \sin \frac{(n+1)\pi y}{b} - (n-1) \sin \frac{(n-1)\pi y}{b}\right)$$

Substitution of these assumed forms for the displacements into the potential energy (Equation 2) yields a function π_p which is thus expressed entirely in terms of trigonometric functions as in Equation 6 (see Appendix A). The integrations required over the area involved can be carried out directly, according to the following:

$$\int_0^a \sin\frac{m\pi x}{a} \sin\frac{r\pi x}{a}\, dx = \frac{a}{2}\delta_{mr}$$

$$\int_0^b \cos\frac{m\pi y}{b} \cos\frac{r\pi y}{b}\, dy = \frac{b}{2}\delta_{mr} \text{ or } b \text{ for } \begin{matrix} m = 0 \\ r = 0 \end{matrix}$$

$$\int_0^a \sin\frac{m\pi x}{a} \cos\frac{r\pi x}{a}\, dx = \frac{2am}{\pi(m^2 - r^2)}\Delta_{mr} \tag{7}$$

$$\int_0^a x \cos\frac{m\pi x}{a} \cos\frac{r\pi x}{a} = \frac{a^2}{4}\delta_{mr} - \frac{a^2}{\pi^2}\left[\frac{1}{(m-r)^2} + \frac{1}{(m+r)^2}\right]\Delta_{mr}$$

$$\text{or } \frac{a^2}{2} \quad \text{for} \quad \begin{matrix} m=0 \\ r=0 \end{matrix}$$

where δ_{mr} is the Kronecker delta, and Δ_{mr} is defined as follows:

$$\Delta_{mr} = \begin{cases} 0; & m + r \text{ even} \\ 1; & m + r \text{ odd} \end{cases}$$

The expression obtained upon substituting the assumed modes for the potential energy (Equation 6) and thus performing the indicated integrations as given above is as shown in Appendix B.

It is again important to recognize that the parameters α_{fx} and α_{fy} contain the locations of the neutral planes (Equation 1). Thus they also are undetermined parameters dependent upon the particular buckling mode; that is,

$$\alpha_{fx} = (\alpha_{fx})_{mn}$$

$$\alpha_{fy} = (\alpha_{fy})_{mn}.$$

Since the energy expression contains cubic terms such as $\alpha_{fxmn} \phi^2$ which are evidently not quadratic forms in the unknowns A_{mn}, B_{mn}, C_{mn}, e_{xmn}, and e_{ymn}, the minimization of the energy will involve a system of nonlinear simultaneous equations in its present form. In order to avoid this difficulty, it is advantageous to define the following additional parameters:

$$H_{mn} = e_{xmn} B_{mn}$$

$$K_{mn} = e_{ymn} C_{mn}.$$

Upon making this addition, the following parameters take on the following forms (Equation 1).

$$\alpha_{fxmn} = \beta_f + (-1)^f \frac{H_{mn}}{B_{mn}}$$

$$\alpha_{fymn} = \beta_f + (-1)^f \frac{K_{mn}}{C_{mn}} \qquad (8)$$

$$\alpha_{fxmn}^2 = \alpha_{fxmn}\alpha_{fx\ell p} = \beta_f^2 + \beta_f(-1)^f \frac{H_{mn}}{B_{mn}} + \beta_f(-1)^f \frac{H_{\ell p}}{B_{\ell p}} + \frac{H_{mn}}{B_{mn}} \frac{H_{\ell p}}{B_{\ell p}}$$

$$\alpha_{fymn}^2 = \alpha_{fymn}\,\alpha_{fy\ell p} = \beta_f^2 + \beta_f(-1)^f \frac{K_{mn}}{C_{mn}} + \beta_f(-1)^f \frac{K_{\ell p}}{C_{\ell p}} + \frac{K_{mn}}{C_{mn}} \frac{K_{\ell p}}{C_{\ell p}}$$

$$\alpha_{fxmn}\,\alpha_{fy\ell p} = \beta_f^2 + \beta_f(-1)^f \frac{H_{mn}}{B_{mn}} + \beta_f(-1)^f \frac{K_{\ell p}}{C_{\ell p}} + \frac{H_{mn}}{B_{mn}} \frac{K_{\ell p}}{C_{\ell p}} .$$

Upon making these substitutions into the expression for the potential energy in Appendix B, the expanded and simplified expression for the potential energy is found in Appendix C in which all terms are quadratic in the unknowns A_{mn}, B_{mn}, C_{mn}, H_{mn}, and K_{mn}.

A linear system of equations of the unknown parameters can be formed when the principle of minimum potential energy is applied.

3.2 PRINCIPLE OF MINIMUM POTENTIAL ENERGY

The principle of minimum potential energy in the variational form is as follows:

$$\delta \pi_p = 0 \tag{9}$$

where π_p is given in Appendix C. After discretization using the Ritz method, the minimum potential energy principle requires that

$$\frac{\partial}{\partial x_i} [\pi_p (x_1, x_2 \ldots x_n)] = 0 \quad i = 1, 2, 3, \ldots n$$

where the x_i are the unknown discrete parameters.

From the expression for the potential energy in Appendix C, there can be formed a set of matrices by rearranging the terms as follows so that the resulting matrices are symmetric about the diagonal of each matrix.

$$\pi_p = X^T DX + \frac{1}{2}\lambda X^T EX \tag{10}$$

$$\pi_p = \frac{1}{2} \underset{5\ell^2}{\begin{Bmatrix} B_{11}\\ B_{12}\\ \vdots\\ B_{\ell\ell}\\ C_{11}\\ C_{12}\\ \vdots\\ C_{\ell\ell}\\ H_{11}\\ H_{12}\\ \vdots\\ H_{\ell\ell}\\ K_{11}\\ K_{12}\\ \vdots\\ K_{\ell\ell}\\ \hline A_{11}\\ A_{12}\\ \vdots\\ A_{\ell\ell} \end{Bmatrix}}^{T} \left[\begin{array}{c|c} D_{11} & D_{12}\\ 4\ell^2 \times 4\ell^2 & 4\ell^2 \times \ell^2\\ \hline D12^T & D22\\ \ell^2 \times 4\ell^2 & \ell^2 \times \ell^2 \end{array}\right] \underset{5\ell^2}{\begin{Bmatrix} B_{11}\\ \vdots \end{Bmatrix}} + \frac{1}{2}\lambda \underset{5\ell^2}{\begin{Bmatrix} B_{11}\\ \vdots \end{Bmatrix}}^{T} \left[\begin{array}{c|c} O & O\\ 4\ell^2 \times 4\ell^2 & 4\ell^2 \times \ell^2\\ \hline O & G\\ \ell^2 \times 4\ell^2 & \ell^2 \times \ell^2 \end{array}\right] \underset{5\ell^2}{\begin{Bmatrix} B_{11}\\ \vdots \end{Bmatrix}}$$

where each vector is of size $5\ell^2$ and each matrix is of total size $5\ell^2 \times 5\ell^2$ which is divided into the submatrices as indicated.

The first matrix (matrix D) contains all the terms associated with the potential energy of the sandwich panel. The second matrix (E) contains all of the terms associated with the work that is applied to the sandwich panel from Appendix C. The only difference is that the applied loads are now written as follows:

$$(\overline{N}_{xo}, \overline{N}_{xB}, \overline{N}_{xy}, \overline{N}_{yo}, \overline{N}_{yB}) = \lambda\ (n_{xo}, n_{xB}, n_{xy}, n_{yo}, n_{yB})$$

where now the submatrix contains the relative magnitudes of the applied loads, thus being able to factor the parameter λ out of the matrix.

To apply the principle of minimum potential energy, it is required now to take the partial derivative of the potential energy against the vector X which contains the unknown discrete parameters. When this is done the resulting expression will be as follows:

$$\frac{\partial \pi_p}{\partial X} = O = DX + \lambda\ EX \tag{11}$$

$$\frac{\partial \pi_p}{\partial X} = \left\{\begin{matrix} O \\ O \\ O \\ O \\ \vdots \\ O \\ \vdots \\ O \end{matrix}\right\}_{5\ell^2} = \left[\begin{array}{c|c} \begin{matrix} D_{11} \\ 4\ell^2 \times 4\ell^2 \end{matrix} & \begin{matrix} D_{12} \\ 4\ell^2 \times \ell^2 \end{matrix} \\ \hline \begin{matrix} D_{12}^{T} \\ \ell^2 \times 4\ell^2 \end{matrix} & \begin{matrix} D_{22} \\ \ell^2 \times \ell^2 \end{matrix} \end{array}\right] \left\{\begin{matrix} B_{11} \\ B_{12} \\ \vdots \\ B_{\ell\ell} \\ C_{11} \\ C_{12} \\ \vdots \\ C_{\ell\ell} \\ H_{11} \\ H_{12} \\ \vdots \\ K_{11} \\ K_{12} \\ \vdots \\ K_{\ell\ell} \\ \hline A_{11} \\ A_{12} \\ \vdots \\ A_{\ell\ell} \end{matrix}\right\}_{5\ell^2} + \lambda \left[\begin{array}{c|c} \begin{matrix} O \\ 4\ell^2 \times 4\ell^2 \end{matrix} & \begin{matrix} O \\ 4\ell^2 \times \ell^2 \end{matrix} \\ \hline \begin{matrix} O \\ \ell^2 \times 4\ell^2 \end{matrix} & \begin{matrix} G \\ \ell^2 \times \ell^2 \end{matrix} \end{array}\right] \left\{\begin{matrix} B_{11} \\ \\ \\ \\ \end{matrix}\right\}_{5\ell^2}$$

Now the expression of the potential energy is in the required form to apply the principle of minimum potential energy. It is now just a matter of solving the system for the unknown parameters:

$$A_{mn},\ B_{mn},\ C_{mn},\ H_{mn},\ K_{mn}\ ,\ \text{and}\ \lambda .$$

This is easily accomplished and will be shown in the next section.

3.3 ASSEMBLY OF EQUATIONS FOR SOLUTION

To easily solve the system of equations in expression (11), the expression must first be rewritten as follows:

$$DX = -\lambda EX \tag{12}$$

$$\left[\begin{array}{c|c} \begin{matrix} D_{11} \\ 4\ell^2 \mathrm{x}\, 4\ell^2 \end{matrix} & \begin{matrix} D_{12} \\ 4\ell^2 \mathrm{x}\ \ell^2 \end{matrix} \\ \hline \begin{matrix} D_{12}^{\ T} \\ \ell^2 \mathrm{x}\, 4\ell^2 \end{matrix} & \begin{matrix} D_{22} \\ \ell^2 \mathrm{x}\, \ell^2 \end{matrix} \end{array}\right] \underset{5\ell^2}{\begin{Bmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{\ell\ell} \\ C_{11} \\ C_{12} \\ \vdots \\ H_{11} \\ H_{12} \\ \vdots \\ H_{\ell\ell} \\ K_{11} \\ K_{12} \\ \vdots \\ K_{\ell\ell} \\ \hline A_{11} \\ A_{12} \\ \vdots \\ A_{\ell\ell} \end{Bmatrix}} = -\lambda \left[\begin{array}{c|c} \begin{matrix} O \\ 4\ell^2 \mathrm{x}\, 4\ell^2 \end{matrix} & \begin{matrix} O \\ 4\ell^2 \mathrm{x}\, \ell^2 \end{matrix} \\ \hline \begin{matrix} O \\ \ell^2 \mathrm{x}\, 4\ell^2 \end{matrix} & \begin{matrix} G \\ \ell^2 \mathrm{x}\, \ell^2 \end{matrix} \end{array}\right] \underset{5\ell^2}{\begin{Bmatrix} B_{11} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{Bmatrix}}$$

To solve this system, one can multiply the submatrices in block form as follows to result in the expressions:

$$
\begin{bmatrix} D_{11} \\ 4\ell^2 \times 4\ell^2 \end{bmatrix}
\overset{X_1}{\underset{4\ell^2}{\begin{bmatrix} B_{11} \\ B_{12} \\ \vdots \\ B_{\ell\ell} \\ \vdots \\ K_{11} \\ K_{12} \\ \vdots \\ K_{\ell\ell} \end{bmatrix}}}
+
\begin{bmatrix} D_{12} \\ 4\ell^2 \times \ell^2 \end{bmatrix}
\overset{X_2}{\underset{\ell^2}{\begin{Bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{\ell\ell} \end{Bmatrix}}}
=
\underset{4\ell^2}{\begin{Bmatrix} O \\ O \\ O \\ \cdot \\ \cdot \\ \cdot \\ O \end{Bmatrix}}
\tag{13}
$$

$$
\begin{bmatrix} D_{12}^{\;T} \\ \ell^2 + 4\ell^2 \end{bmatrix}
\overset{X_1}{\underset{4\ell^2}{\begin{bmatrix} B_{11} \\ B_{12} \\ B_{\ell\ell} \\ \cdot \\ \cdot \\ \cdot \\ K_{11} \\ K_{12} \\ \cdot \\ \cdot \\ \cdot \\ K_{\ell\ell} \end{bmatrix}}}
+
\begin{bmatrix} D_{22} \\ \ell^2 \times \ell^2 \end{bmatrix}
\overset{X_2}{\underset{\ell^2}{\begin{Bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{\ell\ell} \end{Bmatrix}}}
= -\lambda
\begin{bmatrix} G \\ \ell^2 \times \ell^2 \end{bmatrix}
\overset{X_2}{\underset{\ell^2}{\begin{Bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{\ell\ell} \end{Bmatrix}}}
$$

The procedure is to eliminate the vector X_1 in the second expression above. This is accomplished in the following manner. Rearranging the first expression , it can be rewritten as follows:

$$[D_{11}] \{X_1\} = -[D_{12}] \{X_2\}$$

and now solving for X_1 to obtain

$$\{X_1\} = -[D_{11}]^{-1} [D_{12}] \{X_2\}. \tag{14}$$

Now Equation (14) can be rewritten as follows:

$$\{X_1\} = -[P] \{X_2\} \tag{15}$$

where the matrix P is the multiplication of $[D_{11}]^{-1} [D_{12}]$ and therefore has a size of $4\ell^2 \times \ell^2$. Therefore, it follows that:

$$[P] = [D_{11}]^{-1} [D_{12}] \tag{16}$$

for which an equation solver can be used to determine the entries in the matrix P for which the solution is needed. Now the solution for the vector needed in Equation (15) can be substituted into the second expression in Equation (13) to obtain:

$$[D_{12}] (-[P]) \{X_2\} + [D_{22}] \{X_2\} = -\lambda [G] \{X_2\}.$$

The left side of this expression can be rewritten as follows:

$$(-[D_{12}][P] + [D_{22}]) \{X_2\}.$$

The matrix in the above expression is the final stiffness matrix which is as follows:

$$\underset{\ell^2 \times \ell^2}{\left[\ K\ \right]} = -\underset{\ell^2 \times 4\ell^2}{\left[\ D_{12}^T\ \right]}\ \underset{4\ell^2 \times \ell^2}{\left[\ P\ \right]} + \underset{\ell^2 \times \ell^2}{\left[\ D_{22}\ \right]}$$

Therefore, the second expression in Equation (13) can be written as follows:

$$[K] \begin{Bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{\ell\ell} \end{Bmatrix}_{\ell^2} = -\lambda\ [G] \begin{Bmatrix} A_{11} \\ A_{12} \\ \cdot \\ \cdot \\ \cdot \\ A_{\ell\ell} \end{Bmatrix}_{\ell^2}$$

and therefore can be rewritten to obtain:

$$\underset{\ell^2 \times \ell^2}{\left[\ K\ \right]} \begin{Bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{\ell\ell} \end{Bmatrix}_{\ell^2} + \lambda \underset{\ell^2 \times \ell^2}{\left[\ G\ \right]} \begin{Bmatrix} A_{11} \\ A_{12} \\ \vdots \\ A_{\ell\ell} \end{Bmatrix}_{\ell^2} = \{O\}_{\ell^2} \qquad (17)$$

where all the entries in the final stiffness matrix K and the final geometric stiffness matrix G result from the equation for the potential energy in Appendix C.

Finally, the critical loads and the corresponding mode shapes of the panel can be obtained by solving the eigenvalue problem represented by Equation (17).

SECTION 4
SUMMARY OF RESULTS

The procedure which was outlined in Section 3 has been implemented in a computer program. Results have been obtained for several examples of individual critical loads from other articles in order to verify the program's accuracy. Also, critical loads have been calculated for various cases of combined loads and these results have been compared with the existing interactive formulas.

4.1 EXAMPLES

The results of typical calculations performed by the computer program (FSCPAN) to test its validity are presented in Table 1. The results are varied basically due to the type of analysis for clamped sandwich panels performed by the references stated. Each reference takes a slightly varied approach to the analysis.

For these examples, agreement with the previously published results was good in the case of shear loads, but was much lower in the case of compressive loads. The critical loads calculated for compression are much lower than those cited because the modes given do not uncouple. Therefore, it is not possible to solve the equations in closed form as in the simple support case in Reference 8.

Convergence of these examples is limited to eight terms where $\ell = 8$ in Section 3 (Equation 6) of this report. Using eight terms, the amount of computer space needed is approximately one-half the available computer capacity on the CYBER 74 at Wright-Patterson Air Force Base. If needed, more terms could be used, but the amount of space needed increases drastically as more terms are added to the expression (Equation 6).

As for pure edgewise bending, there were no previous results found to verify the computer program. The only results available are those of the interaction formulas which will be examined more closely later.

TABLE 1

VERIFICATION RESULTS

Construction	Loading	E_1 E_2	ν_1 ν_2	t_1 t_2	G_{cxz} G_{cyz}	t_c	a b	Critical Load	Ref	Reference Results
Sandwich	Compr.	9.5E6 10.6E6	.3 .25	.021 .1	35000 21000	.5	35 21	N_x=7377.06	6	8109.10
Sandwich	Compr.	9.5E6 9.5E6	.3 .3	.05 .1	19000 21000	.75	40 20	N_x=15267.04	6	16156.70
Sandwich	Compr.	10.6E6 10.6E6	.25 .25	.05 .1	35000 30000	.75	20. 24.	N_x=22195.90	6	24016.06
Sandwich	Shear	30.E6 30.E6	.3 .3	.0075 .0075	38000 38000	.25	40 20	N_{xy}=1785.70	2	1762
Thin Plate	Shear	30.E6 30.E6	.3 .3	.05 .05	∞ ∞	0	10 10	N_{xy}=3998.54	2	3630
Thin Plate	Shear	1.E7 1.E7	.3 .3	.05 .05	∞ ∞	0	40 40	N_{xy}=83.48	7	83.09
Thin Plate	Shear	1.E7 1.E7	.3 .3	.1 .1	∞ ∞	0	35 35	N_{xy}=869.72	7	868.25

4.2 LEVY ANALYSIS

Critical loads calculations for typical isotropic panels subjected to compressive loads are presented in Figure 4. Samuel Levy (Reference 1) has determined a relation between the critical stress ratios and the length to width ratio for isotropic flat panels subjected to an edgewise compressive load. His analysis starts with the case of the simple support panel in which the displacements of the panel are assumed to be zero; then by revising this analysis he is able to adjust the coefficients so that the slope of the clamped edges is also zero. His analysis leads to an infinite series solution for which convergence is extremely rapid. In Figure 4 his results are plotted as indicated for the critical stress ratio

$$\frac{(\sigma_x)_{cr} \; b^2 h}{\pi^2 D}$$

where

$$D = \frac{Eh^3}{12(1 - \upsilon^2)} \quad .$$

To verify the analysis of this report, three panels were used to obtain results similar to that of Levy. These panels are presented in Table 2. The panels presented were used in this analysis, computer program (FSCPAN) and MIL-HDBK-23A analysis for clamped panels. As can be seen in Figure 4 the results of FSCPAN are higher than the Levy analysis due to the fact that the FSCPAN results are not an infinite series solution, but are lower than that of the Handbook analysis for the length to width ratios of the three panels.

For the Handbook the results are exact solutions for the three panels, but the FSCPAN results are for eight terms of the infinite series needed to solve the system. With more terms, the results will be much lower than those presented and will eventually become much closer to the Levy results.

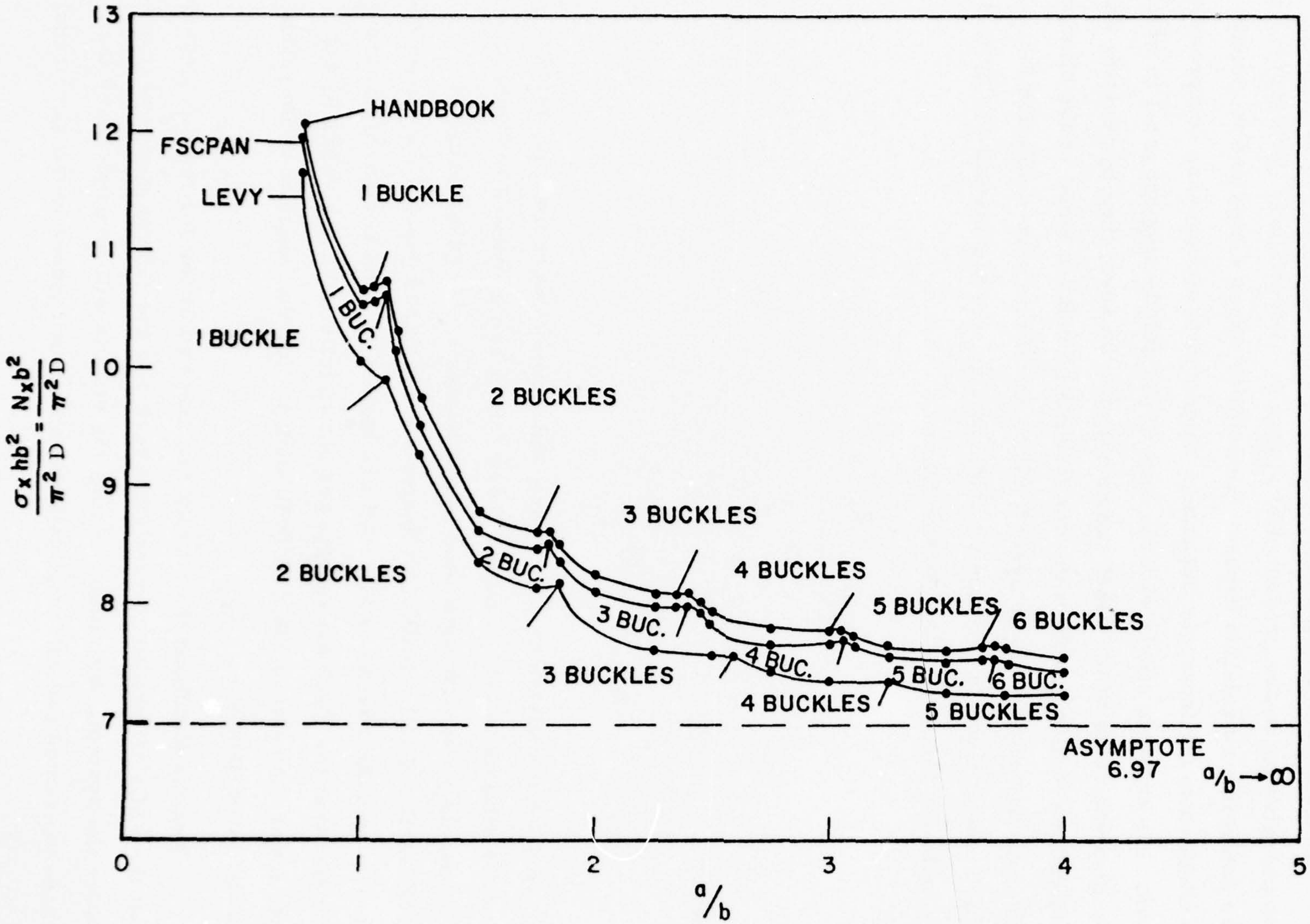

Figure 4. Levy Critical Stress Ratio.

TABLE 2. LEVY PLATES

	Plate A	Plate B	Plate C
E_1	9.5E6	10.6E6	29.4E6
E_2	9.5E6	10.6E6	29.4E6
t_1	.1	.15	.02
t_2	.1	.15	.02
ν_1	.3	.3	.3
ν_2	.3	.3	.3
G_{cxz}	∞	∞	∞
G_{cyz}	∞	∞	∞
t_c	o	o	o
b	20.	20.	20.

a goes from 15 to 80

4.3 INTERACTION FORMULAS

A number of interaction formulas have been proposed in order to facilitate the analysis of the stability for cases of combined loadings for sandwich panels as given in Reference 6. The purpose of the interaction formulas is to account for the combined effects on the panel of various applied loads by the comparison of the intensity of each load to the critical value of the same load acting alone on the panel. Each ratio is weighted by means of an exponent associated with the particular load type being examined. For example

$$R_1^{e_1} + R_2^{e_2} = \text{const.}$$

where

$$R_i = \frac{\bar{N}_i}{N_{icr}} .$$

The e_i are constant exponents, and the index i is associated with the type of loading being examined.

Interaction formulas are attractive in that any number of load cases can be analyzed once the individual critical loads for the panel are determined. The problem is that these formulas represent approximations to an exact analysis of the stability under the combined loadings. Interaction formulas for the cases of edgewise compression with edgewise shear, edgewise compression with edgewise bending, and edgewise shear with edgewise bending have been evaluated for the sandwich panels in Table 3 during this analysis.

4.3.1 Edgewise Shear and Compression

The interaction formula for axial compression and edgewise shear is of the following form:

TABLE 3. PHYSICAL DATA FOR COMBINED-LOAD EXAMPLES

	Panel 1	Panel 2
E_1	9.5×10^6	10.6×10^6
E_2	9.5×10^6	10.6×10^6
t_1	.021	.050
t_2	.021	.050
ν_1	.30	.30
ν_2	.30	.30
G_{cxz}	19000.	30000.
G_{cyz}	19000.	30000.
t_c	.181	.25
a	20.	20.
b	20.	36.

$$\frac{\overline{N}_x}{\overline{N}_{xcr}} + \frac{\overline{N}_y}{\overline{N}_{ycr}} + \left(\frac{N_{xy}}{\overline{N}_{xycr}}\right)^2 = 1.$$

For simplicity of this analysis and for the purpose of graphing, the above formula has been modified to the following:

$$\frac{\overline{N}_x}{\overline{N}_{xcr}} + \left(\frac{\overline{N}_{xy}}{\overline{N}_{xycr}}\right)^2 = 1 \qquad (18)$$

The left-hand side of Equation (18) has been evaluated for several load conditions. The resulting values of the equation have been graphed and are presented in Figure 5 for the two panels. The formula is reasonably accurate for most of the cases considered. For Panel 1 the estimate (Equation 18) is a nonconservative estimate in that the true values for the panel are less than the estimate below.

4.3.2 Edgewise Bending and Compression

The interaction formula for combined edgewise bending and axial compression loads is given as follows:

$$\frac{\overline{N}_x}{\overline{N}_{xcr}} + \left(\frac{\overline{N}_{xB}}{\overline{N}_{xBcr}}\right)^{3/2} = 1. \qquad (19)$$

Here $\overline{N}_{xB}$ is a pure-bending load as shown in Figure 1. The left-hand side of the formula has been evaluated for several load conditions. The resulting values of the equation have been graphed, and are presented in Figure 6 for the two panels. The formula is reasonably accurate for most of the cases, but has the greater discrepancy for Panel 1. For both panels, the estimate (Equation 19) is a conservative estimate for the true values of the edgewise compression and edgewise bending for the panel.

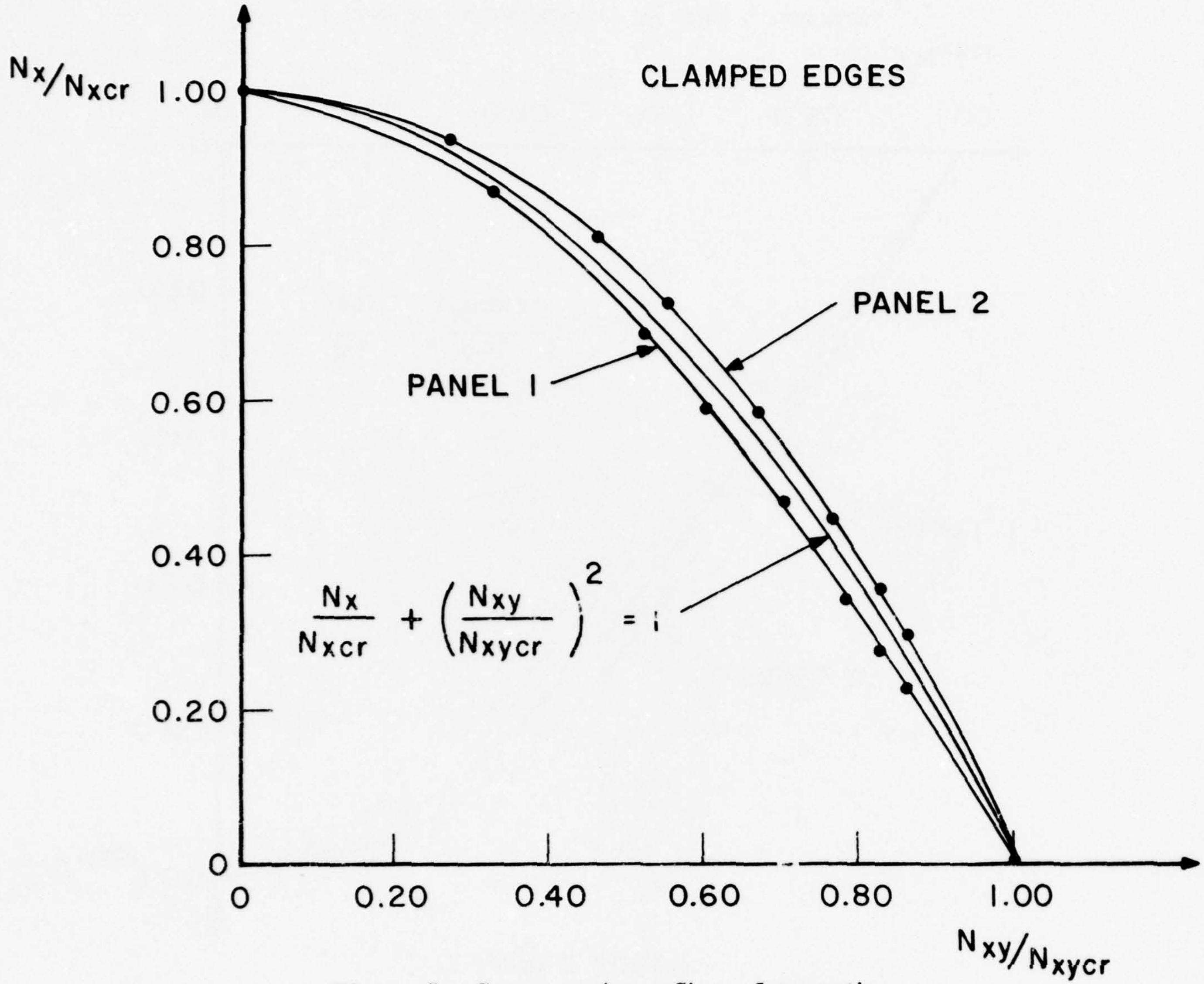

Figure 5. Compression - Shear Interaction.

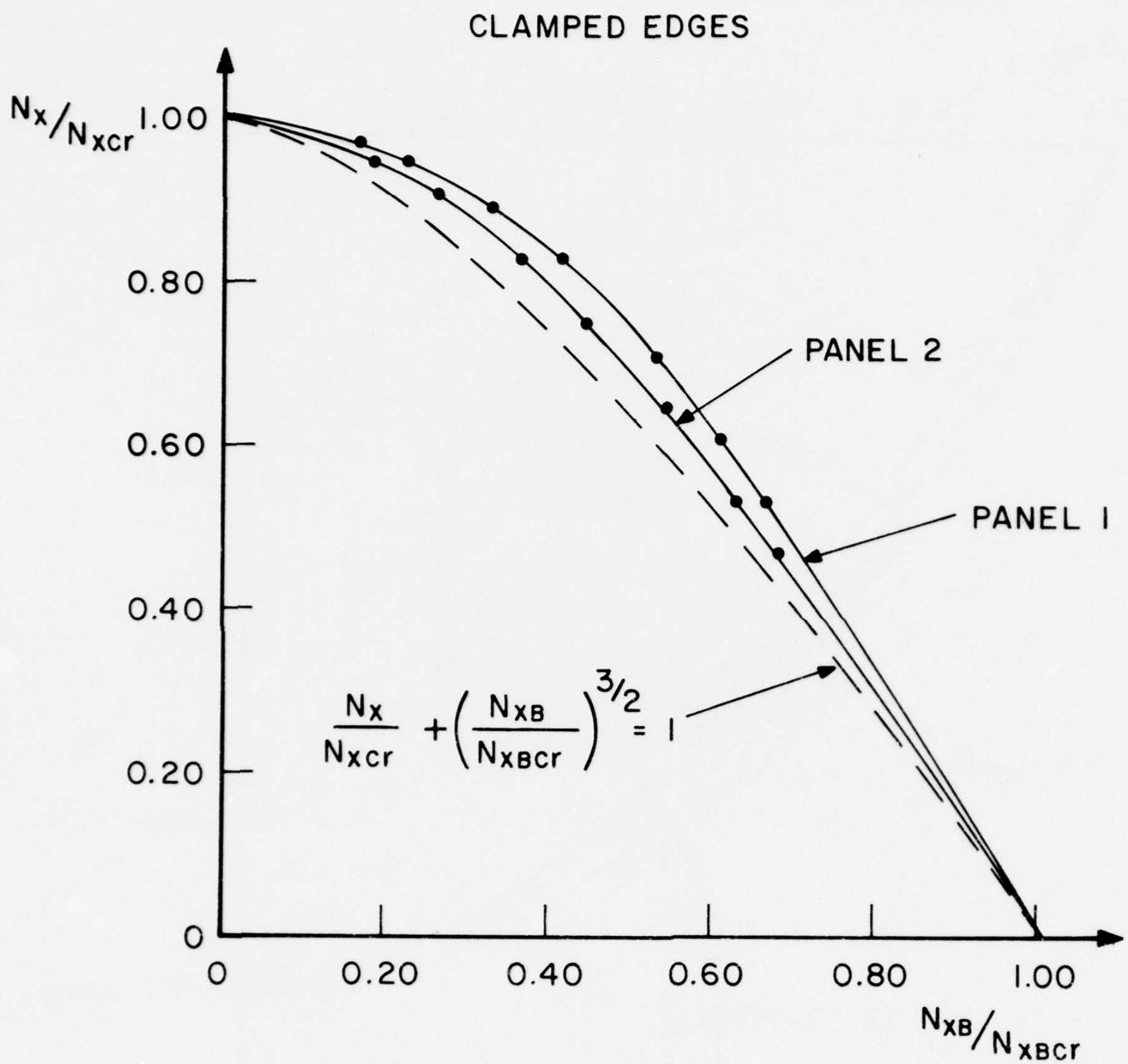

Figure 6. Compression - Bending Interaction.

4.3.3 Edgewise Bending and Shear

The interaction formula for the evaluation of critical combined edgewise bending and shear loads is as follows:

$$\left(\frac{\bar{N}_{xB}}{\bar{N}_{xBcr}}\right)^2 + \left(\frac{\bar{N}_{xy}}{\bar{N}_{xycr}}\right)^2 = 1. \tag{20}$$

Values of the left-hand side of the formula have been evaluated for the two panels. These values have been graphed and are shown in Figure 7. The expression is relatively accurate for the two panels. For both panels the estimate (Equation 20) is a nonconservative estimate of the true values for the edgewise bending and shear for the panels. For Panel 1 the estimate is almost the true values for the panel in that the graph for Panel 1 almost lies on top of the formula. For Panel 2 the estimate is much more nonconservative, in that the true values for the panel lie much farther below the estimate for the values.

4.4 OTHER EXAMPLES

Out of curiosity, the analysis led to the three panels listed in Table 4. For each of the three panels several values of the length to width ratio were considered. Results for edgewise compression only were computed for the three panels with the computer program (FSCPAN) and the Handbook. As can be seen in Figures 8, 9, and 10, the critical edgewise compression load obtained for each of the three panels is lower by the analysis presented in this report (FSCPAN) than by the analysis presented in the Handbook (Reference 6).

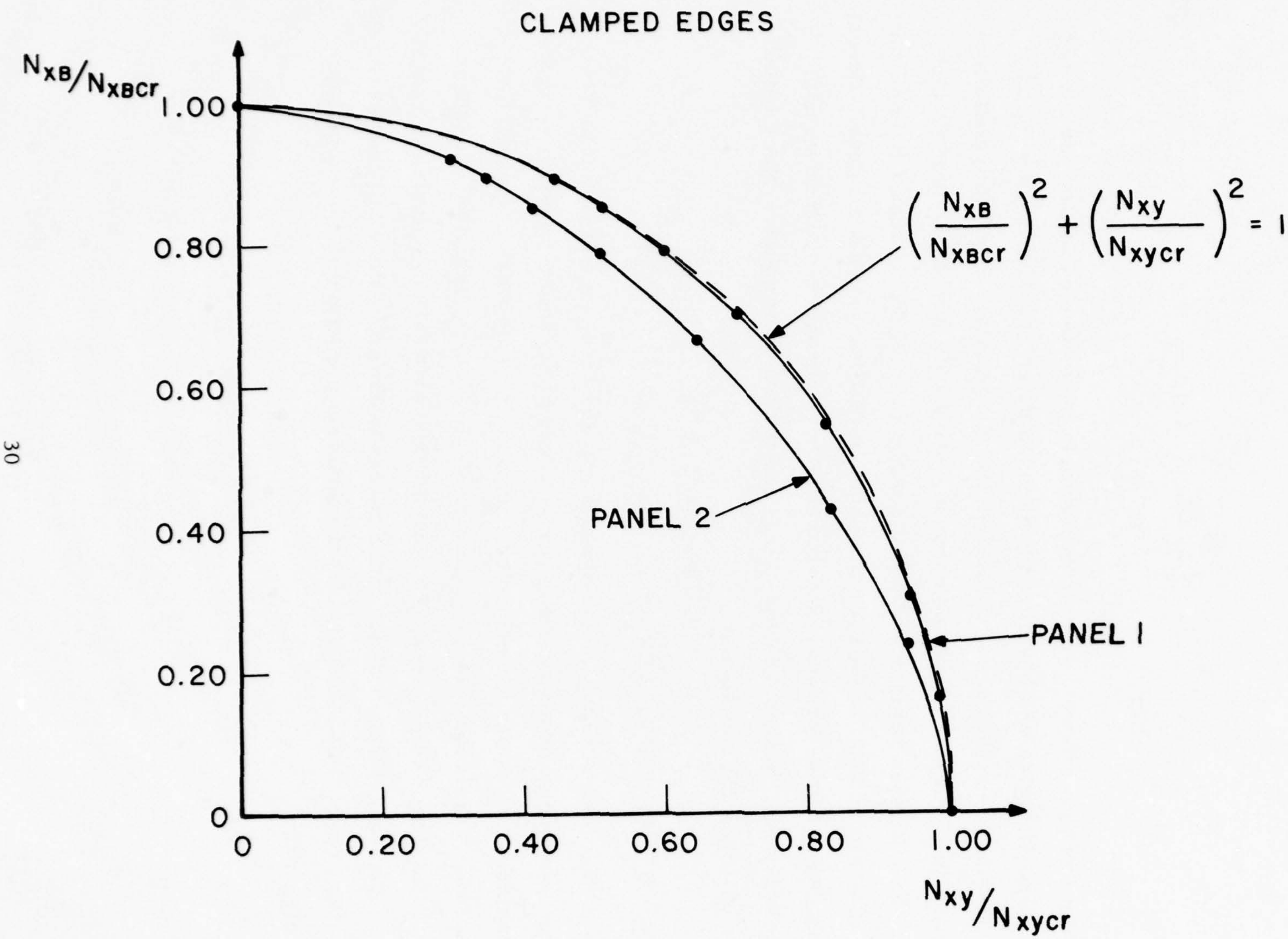

Figure 7. Bending-Shear Interaction.

TABLE 4. SANDWICH PANELS

	Panel A	Panel B	Panel C
E_1	10.6E6	10.6E6	10.6E6
E_2	10.6E6	10.6E6	10.6E6
t_1	.05	.125	.025
t_2	.05	.125	.025
ν_1	.3	.3	.3
ν_2	.3	.3	.3
G_{cxz}	21000	21000	21000
G_{cyz}	21000	21000	21000
t_c	.5	.5	.5
b	20.	20.	20.

a 15 → 60

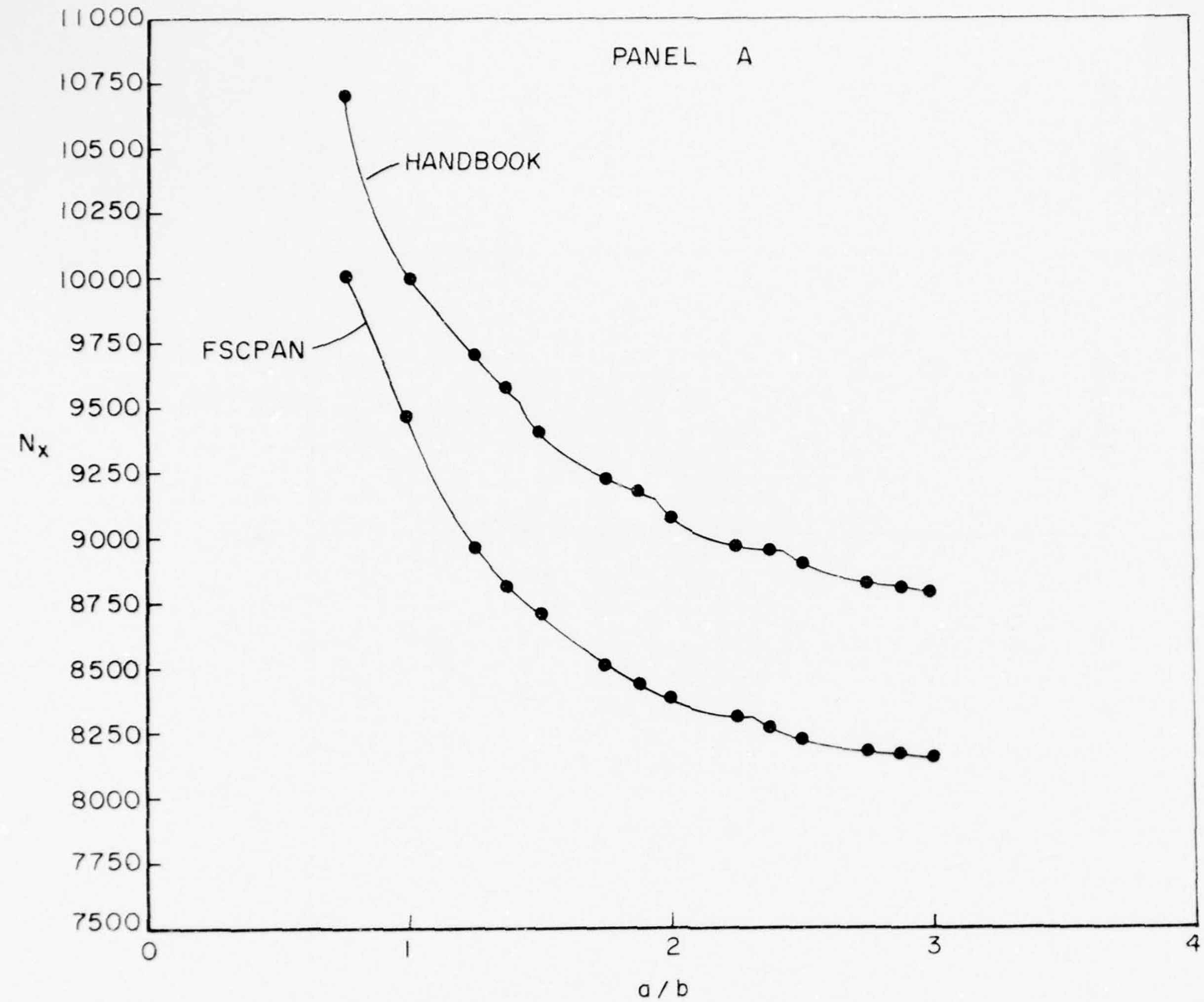

Figure 8. Handbook - FSCPAN Comparison.

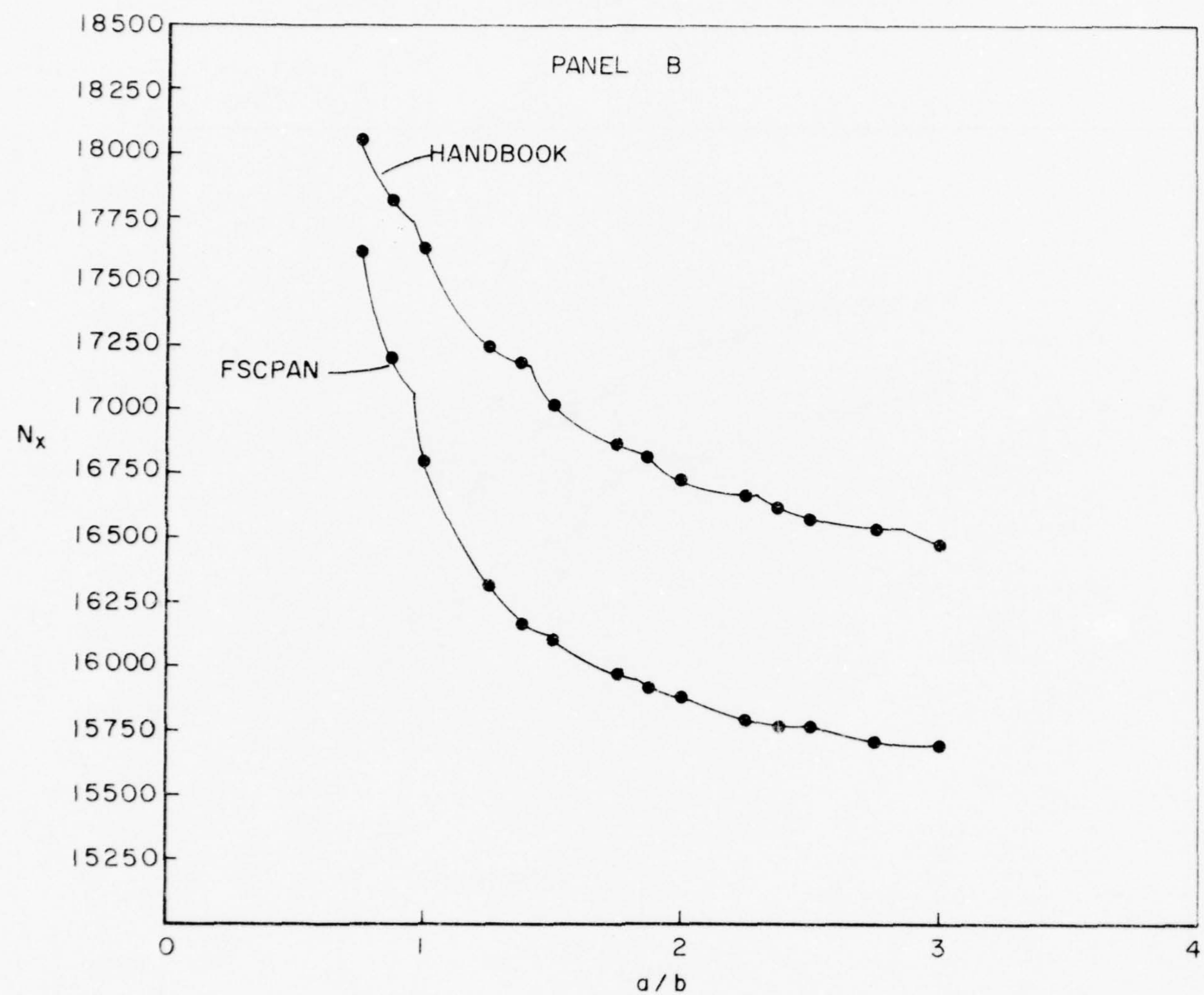

Figure 9. Handbook - FSCPAN Comparison.

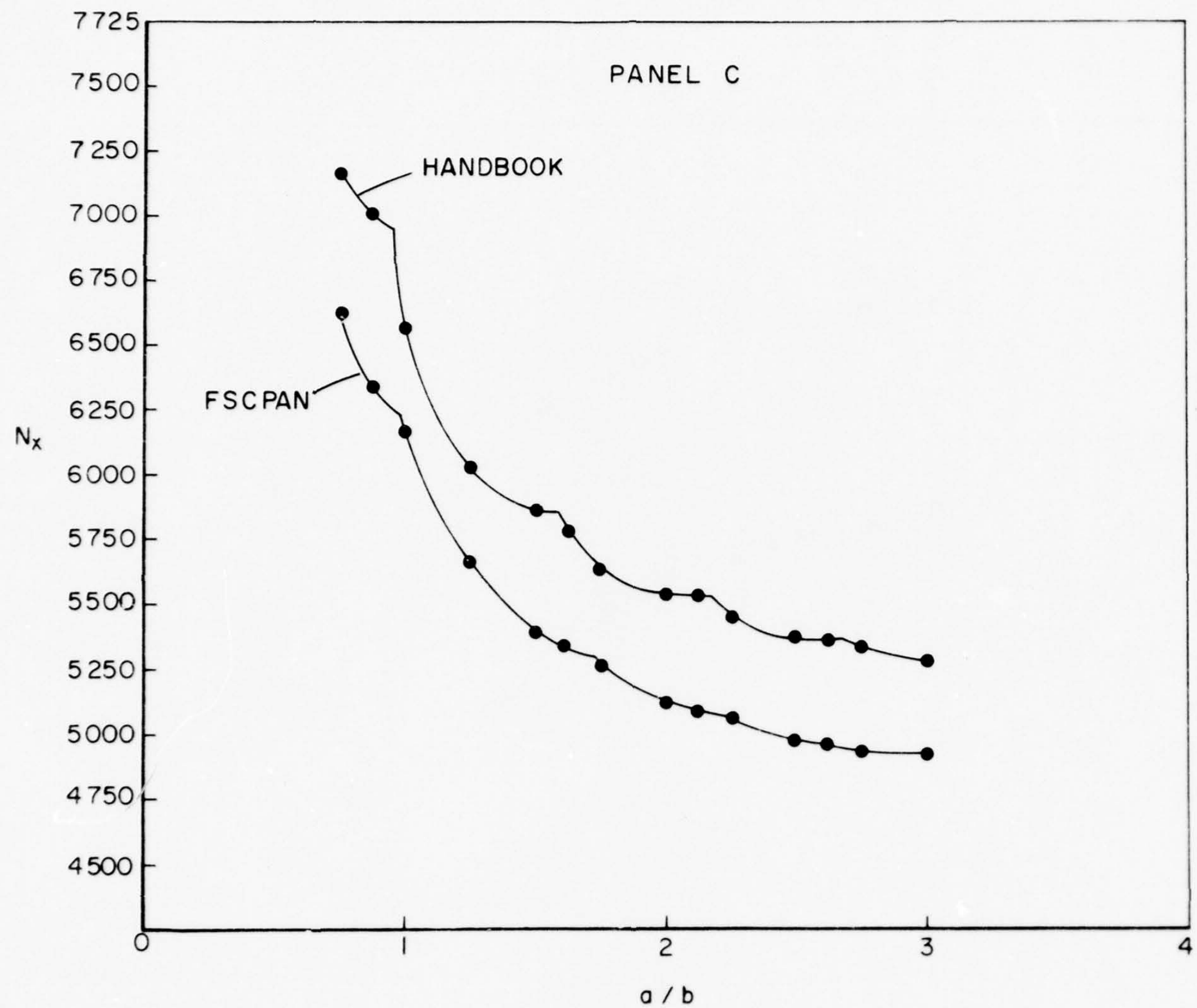

Figure 10. Handbook - FSCPAN Comparison.

4.5 DISCUSSION

In this section, various calculations concerning the stability of a clamped sandwich panel under combined modes of loading and single loading have been presented along with conclusions concerning the effectiveness of typical interaction formulas. These results indicate the versatility of the analysis procedure described herein, and illustrate the character of some of the effects due to combined edge loadings to clamped panels.

The results given in Paragraph 4.2 indicate that the infinite series solution to the analysis agrees with that of Levy for the type of analysis he has performed. Also, for the eight-term solution used for this analysis, the results for an edgewise compressive load applied to an isotropic panel are much lower than those results as presented in the analysis in Reference 6. The results given in Paragraph 4.4 also indicate that the analysis presented herein for an edgewise compressive load yields lower values than those values as presented in Reference 6.

Results given for the various interaction formulas indicate that the formula proposed[6] for edgewise compression and edgewise bending (Equation 19) may be of some use for design purposes, since this formula yields rather conservative results for all cases considered herein. For the case of combined edgewise compression and shear loads (Equation 18) the formula is conservative for a rectangular panel, but is nonconservative for a square panel. As for predictability, the formula will probably depend on the type of panel being considered and the length-width ratio being considered. As for the edgewise bending and shear interaction formula (Equation 20), the results obtained will be nonconservative for the design of a sandwich panel. The best results will be obtained for a square sandwich panel as shown in Figure 7. As for a rectangular panel the formula is very nonconservative and may possibly lead to a rather large error in the true values when predicting the combined load values.

SECTION 5
SUMMARY AND CONCLUSIONS

An analysis has been presented for the prediction of the instability of a flat, clamped, rectangular sandwich panel loaded by combinations of biaxial edgewise compression, biaxial edgewise bending, and edgewise shear loads. The model has been formulated in terms of the total potential energy of the panel, which is composed of internal elastic strain energy and the potentials of the combined applied loads. The Ritz method is used to transform the energy function in terms of continuous field variables into a quadratic function of discrete parameters. Transforming this function into a matrix form and applying the principle of minimum potential energy results in a generalized discrete eigenvalue problem. Solution to this eigenvalue problem is done by standard methods. The finite number of eigenvalues obtained in the solution approximate the lowest value of the infinite number of critical loads to find the critical buckling load, while the eigenvectors approximate the buckled mode shapes that the panel will assume corresponding to the given eigenvalue.

The accuracy of this analysis has been demonstrated by the comparison with results available in the literature for a rather limited number of examples. Also, a limited study has been made to look at the applicability of several typical interactive formulas which are commonly used in the design of sandwich panels. From the present analysis, comparisons of the interactive formulas indicates that some of these formulas yield nonconservative results for certain combinations of loadings and geometries, and should definitely be used with extreme caution.

A computer program has been developed to implement the combined-loads instability analysis presented herein. The program is operational only by batch mode processing. For the program to operate interactively, the number of terms in the expression (Appendix A) would have to be so

drastically reduced that the results would not be very useful to obtain the true value as can be approximated with the batch-mode with eight terms. A detailed description of the computer program and its usage are contained in a companion report.[9]

In conclusion, the sandwich panel instability analysis described herein represents a useful and somewhat accurate tool for the design of light-weight, high-performance structural components having clamped edges which is the case in many physical structures. In addition, the method of analysis provides a suitable starting point from which many other general types of sandwich construction can be considered.

APPENDIX A

POTENTIAL ENERGY IN INTEGRAL FORM

APPENDIX A

POTENTIAL ENERGY IN INTEGRAL FORM

This Appendix presents the expansion of the Equations (5 and 6) and the symmetry needed for the expansion of Equation (2).

$$\pi_p = \sum_m\sum_n\sum_r\sum_s \left\{ \sum_{i=1}^{2} \frac{1}{2}\frac{E_i}{\lambda_i} \left[\alpha_{ix}^2 t_i \int_0^b\int_0^a B_{mn}B_{rs}\frac{\pi^2}{a^2} \left((m+1)^2\cos\frac{(m+1)\pi x}{a} - (m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right)\right.\right.$$

$$\left((r+1)^2\cos\frac{(r+1)\pi x}{a} - (r-1)^2\cos\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b} - \cos\frac{(s+1)\pi y}{b}\right)dxdy$$

$$+\alpha_{ix}^2 t_i\left(\frac{1-\nu_i}{2}\right)\int_0^b\int_0^a B_{mn}B_{rs}\frac{\pi^2}{b^2}\left((m+1)\sin\frac{(m+1)\pi x}{a} - (m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b} - (n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a} - (r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b} - (s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+2\alpha_{ix}\alpha_{ij}t_i\nu_i\left(\frac{1}{2}\int_0^b\int_0^a B_{mn}C_{rs}\frac{\pi^2}{ab}\left((m+1)^2\cos\frac{(m+1)\pi x}{a} - (m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b} - \cos\frac{(n+1)\pi y}{b}\right)\right.$$

$$\left(\cos\frac{(r-1)\pi x}{a} - \cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b} - (s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a C_{mn}B_{rs}\frac{\pi^2}{ab}\left(\cos\frac{(m-1)\pi x}{a} - \cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b} - (n-1)^2\cos\frac{(n-1)\pi y}{b}\right)$$

$$\left.\left((r+1)^2\cos\frac{(r+1)\pi x}{a} - (r-1)^2\cos\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b} - \cos\frac{(s+1)\pi y}{b}\right)dxdy\right)$$

$$+\alpha_{ix}\alpha_{iy}t_i(1-\nu_i)\left(\frac{1}{2}\int_0^b\int_0^a B_{mn}C_{rs}\frac{\pi^2}{ab}\left((m+1)\sin\frac{(m+1)\pi x}{a} - (m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b} - (n-1)\sin\frac{(n-1)\pi y}{b}\right)\right.$$

$$\left((r+1)\sin\frac{(n+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\!\!\int_0^a C_{mn}B_{rs}\frac{\pi^2}{ab}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy\Bigg)$$

$$+\alpha_{iy}^2 t_i\int_0^b\!\!\int_0^a C_{mn}C_{rs}\frac{\pi^2}{b^2}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\alpha_{iy}^2 t_i\left(\frac{1-\nu_i}{2}\right)\int_0^b\!\!\int_0^a C_{mn}C_{rs}\frac{\pi^2}{a^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\alpha_{ix}t_i^2\Bigg(\frac{1}{2}\int_0^b\!\!\int_0^a B_{mn}A_{rs}\frac{\pi^3}{a^3}\left((m+1)^2\cos\frac{(m+1)\pi x}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)^2\cos\frac{(r+1)\pi x}{a}-(r-1)^2\cos\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\!\!\int_0^a A_{mn}B_{rs}\frac{\pi^3}{a^3}\left((m+1)^2\cos\frac{(m+1)\pi x}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)^2\cos\frac{(r+1)\pi x}{a}-(r-1)^2\cos\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy\Bigg)$$

$$+\alpha_{ix}\nu_i t_i^2\left(\frac{1}{2}\int_0^b\int_0^a B_{mn}A_{rs}\frac{\pi^2}{ab^2}\left((m+1)^2\cos\frac{(m+1)\pi x}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)\right.$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}B_{rs}\frac{\pi^3}{ab^2}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\right)$$

$$\left.\left((r+1)^2\cos\frac{(r+1)\pi x}{a}-(r-1)^2\cos\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy\right)$$

$$+\alpha_{ix}t_i^2(1-\nu_i)\left(\frac{1}{2}\int_0^b\int_0^a B_{mn}A_{rs}\frac{\pi^3}{ab^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)\right.$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}B_{rs}\frac{\pi^3}{ab^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left.\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy\right)$$

$$+\alpha_{iy}t_i^2\left(\frac{1}{2}\int_0^b\int_0^a C_{mn}A_{rs}\frac{\pi^3}{b^3}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\right)\right.$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}C_{rs}\quad\frac{\pi^3}{b^3}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\right)$$

$$\bigg(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\bigg)\bigg((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\bigg)dxdy\Bigg)$$

$$+q_{iy}t_i^2\nu_i\Bigg(\frac{1}{2}\int_0^b\int_0^a C_{mn}A_{rs}\frac{\pi^3}{ba^2}\bigg(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\bigg)\bigg((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\bigg)$$

$$\bigg((r+1)^2\cos\frac{(r+1)\pi x}{a}-(r-1)^2\cos\frac{(r-1)\pi x}{a}\bigg)\bigg(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\bigg)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}C_{rs}\frac{\pi^3}{ba^2}\bigg((m+1)^2\cos\frac{(m+1)\pi x}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\bigg)\bigg(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\bigg)$$

$$\bigg(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\bigg)\bigg((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\bigg)dxdy\Bigg)$$

$$+q_{iy}t_i^2(1-\nu_i)\Bigg(\frac{1}{2}\int_0^b\int_0^a C_{mn}A_{rs}\frac{\pi^3}{ba^2}\bigg((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\bigg)\bigg((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\bigg)$$

$$\bigg((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\bigg)\bigg((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\bigg)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}C_{rs}\frac{\pi^3}{ba^2}\bigg((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\bigg)\bigg((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\bigg)$$

$$\bigg((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\bigg)\bigg((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\bigg)dxdy\Bigg)$$

$$+\frac{1}{3}t_i^3\int_0^b\int_0^a A_{mn}A_{rs}\frac{\pi^4}{a^4}\bigg((m+1)^2\cos\frac{(m+1)\pi x}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\bigg)\bigg(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\bigg)$$

$$\bigg((r+1)^2\cos\frac{(r+1)\pi x}{a}-(r-1)^2\cos\frac{(r-1)\pi x}{a}\bigg)\bigg(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\bigg)dxdy$$

$$+\frac{2}{3}t_i^3\nu_i\int_0^b\int_0^a A_{mn}A_{rs}\,\frac{\pi^4}{a^2b^2}\left((m+1)^2\cos\frac{(m+1)\pi y}{a}-(m-1)^2\cos\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{3}t_i^3\int_0^b\int_0^a A_{mn}A_{rs}\,\frac{\pi^4}{b^4}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)^2\cos\frac{(n+1)\pi y}{b}-(n-1)^2\cos\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)^2\cos\frac{(s+1)\pi y}{b}-(s-1)^2\cos\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{2}{3}t_i^3(1-\nu_i)\int_0^b\int_0^a A_{mn}A_{rs}\,\frac{\pi^4}{a^2b^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\frac{(s-1)\pi y}{b}\right)dxdy\Bigg]$$

$$+\frac{t_c}{2}G_{cxz}\int_0^b\int_0^a B_{mn}B_{rs}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy$$

$$+t_cG_{cxz}\left(\frac{1}{2}\int_0^b\int_0^a B_{mn}A_{rs}\,\frac{\pi}{a}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)\right.$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}B_{rs}\,\frac{\pi}{a}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy\Bigg)$$

$$+\frac{t_c}{2}G_{cxz}\int_0^b\int_0^a A_{mn}A_{rs}\frac{\pi^2}{a^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right)dxdy$$

$$+\frac{t_c}{2}G_{cyz}\int_0^b\int_0^a C_{mn}C_{rs}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+t_cG_{cyz}\Bigg(\frac{1}{2}\int_0^b\int_0^a C_{mn}A_{rs}\;\frac{\pi}{b}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}C_{rs}\;\frac{\pi}{b}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\,dxdy\right)$$

$$+\frac{t_c}{2}G_{cyz}\int_0^b\int_0^a A_{mn}A_{rs}\frac{\pi^2}{b^2}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right)dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a\left(N_{xo}+N_{xB}\left(1-\frac{2y}{b}\right)\right) A_{mn}A_{rs}\frac{\pi^2}{a^2}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)$$

$$\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right) dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a\left(N_{yo}+N_{yB}\left(1-\frac{2x}{a}\right)\right) A_{mn}A_{rs}\frac{\pi^2}{b^2}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right) dxdy$$

$$+N_{xy}\left(\frac{1}{2}\int_0^b\int_0^a A_{mn}A_{rs}\frac{\pi^2}{ab}\left((m+1)\sin\frac{(m+1)\pi x}{a}-(m-1)\sin\frac{(m-1)\pi x}{a}\right)\left(\cos\frac{(n-1)\pi y}{b}-\cos\frac{(n+1)\pi y}{b}\right)\right.$$

$$\left(\cos\frac{(r-1)\pi x}{a}-\cos\frac{(r+1)\pi x}{a}\right)\left((s+1)\sin\frac{(s+1)\pi y}{b}-(s-1)\sin\frac{(s-1)\pi y}{b}\right) dxdy$$

$$+\frac{1}{2}\int_0^b\int_0^a A_{mn}A_{rs}\frac{\pi^2}{ab}\left(\cos\frac{(m-1)\pi x}{a}-\cos\frac{(m+1)\pi x}{a}\right)\left((n+1)\sin\frac{(n+1)\pi y}{b}-(n-1)\sin\frac{(n-1)\pi y}{b}\right)$$

$$\left.\left.\left((r+1)\sin\frac{(r+1)\pi x}{a}-(r-1)\sin\frac{(r-1)\pi x}{a}\right)\left(\cos\frac{(s-1)\pi y}{b}-\cos\frac{(s+1)\pi y}{b}\right) dxdy\right)\right\}$$

APPENDIX B

POTENTIAL ENERGY IN EVALUATED FORM

APPENDIX B

POTENTIAL ENERGY IN EVALUATED FORM

This Appendix presents the expansion of the Equation (7) into Equation (2) where

$$\gamma = \begin{cases} 2 & n \neq 1 \\ 3 & n = 1 \end{cases} \quad \text{and} \quad \theta = \begin{cases} 2 & m \neq 1 \\ 3 & m = 1 \end{cases} .$$

$$\pi_p = \sum_m \sum_n \Bigg[\sum_{i=1}^{2} \frac{1}{2} \frac{E_i}{\lambda_i} \Bigg\{ t_i \alpha_{ix}^{\,2} \frac{\pi^2 b}{4a} \Bigg((m+1)^4 \Big(\gamma B^2_{mn} - \gamma B_{mn} B_{m+2,n} + B_{mn} B_{m+2,n+2} - B_{mn} B_{m,n-2}$$

$$- B_{mn} B_{m,n+2} + B_{mn} B_{m+2,n-2} \Big)$$

$$+ (m-1)^4 \Big(\gamma B^2_{mn} - \gamma B_{mn} B_{m-2,n} - B_{mn} B_{m,n+2} + B_{mn} B_{m-2,n+2} - B_{mn} B_{m,n-2} + B_{mn} B_{m-2,n-2} \Big) \Bigg)$$

$$+ t_i \left(\frac{1 - \upsilon_i}{2} \right) \alpha^2_{ix} \frac{\pi^2 a}{4b} \Bigg((m+1)^2 (n+1)^2 \Big(B^2_{mn} - B_{mn} B_{m,n+2} - B_{mn} B_{m+2,n} + B_{mn} B_{m+2,n+2} \Big)$$

$$+ (m+1)^2 (n-1)^2 \Big(B^2_{mn} + B_{mn} B_{m+2,n-2} - B_{mn} B_{m+2,n} - B_{mn} B_{m,n-2} \Big)$$

$$+ (m-1)^2 (n+1)^2 \Big(B^2_{mn} - B_{mn} B_{m-2,n} + B_{mn} B_{m-2,n+2} - B_{mn} B_{m,n+2} \Big)$$

$$+ (m-1)^2 (n-1)^2 \Big(B^2_{mn} + B_{mn} B_{m-2,n-2} - B_{mn} B_{m-2,n} - B_{mn} B_{m,n-2} \Big) \Bigg)$$

$$+ t_i (1 + \upsilon_i) \alpha_{ix} \alpha_{iy} \frac{\pi^2}{8} \Bigg((m+1)^2 (n-1)^2 \Big(2B_{mn} C_{mn} + B_{mn} C_{m+2,n-2} - B_{mn} C_{m,n-2} - B_{mn} C_{m+2,n}$$

$$+ C_{mn} B_{m+2,n-2} - C_{mn} B_{m,n-2} - C_{mn} B_{m+2,n} \Big)$$

$$+ (m+1)^2 (n+1)^2 \Big(2B_{mn} C_{mn} + B_{mn} C_{m+2,n+2} - B_{mn} C_{m+2,n} - B_{mn} C_{m,n+2} + C_{mn} B_{m+2,n+2}$$

$$- C_{mn} B_{m+2,n} - C_{mn} B_{m,n+2} \Big)$$

$$+ (m-1)^2 (n-1)^2 \Big(2B_{mn} C_{mn} + B_{mn} C_{m-2,n-2} - B_{mn} C_{m,n-2} - B_{mn} C_{m-2,n} + C_{mn} B_{m-2,n-2}$$

$$- C_{mn} B_{m-2,n} - C_{mn} B_{m,n-2} \Big)$$

$$+ (m-1)^2 (n+1)^2 \Big(2B_{mn} C_{mn} + B_{mn} C_{m-2,n+2} - B_{mn} C_{m-2,n} - B_{mn} C_{m,n+2} + C_{mn} B_{m-2,n+2}$$

$$- C_{mn} B_{m-2,n} - C_{mn} B_{m,n+2} \Big) \Big)$$

$$+ t_i \alpha_{iy}^2 \frac{\pi^2 a}{4b} \Big((n+1)^4 \Big(8\, C_{mn}^2 - 8\, C_{mn} C_{m,n+2} - C_{mn} C_{m-2,n} + C_{mn} C_{m-2,n+2} - C_{mn} C_{m+2,n} + C_{mn} C_{m+2,n+2} \Big)$$

$$+ (n-1)^4 \Big(8\, C_{mn}^2 - 8\, C_{mn} C_{m,n-2} + C_{mn} C_{m-2,n-2} - C_{mn} C_{m-2,n} + C_{mn} C_{m+2,n-2} - C_{mn} C_{m+2,n} \Big) \Big)$$

$$+ t_i \alpha_{iy}^2 \left(\frac{1-\upsilon_i}{2} \right) \frac{\pi^2 b}{4a} \Big((m+1)^2 (n+1)^2 \Big(C_{mn}^2 - C_{mn} C_{m,n+2} - C_{mn} C_{m+2,n} + C_{mn} C_{m+2,n+2} \Big)$$

$$+ (m+1)^2 (n-1)^2 \Big(C_{mn}^2 - C_{mn} C_{m,n-2} - C_{mn} C_{m+2,n} + C_{mn} C_{m+2,n-2} \Big)$$

$$+ (m-1)^2 (n+1)^2 \Big(C_{mn}^2 - C_{mn} C_{m-2,n} - C_{mn} C_{m,n+2} + C_{mn} C_{m-2,n+2} \Big)$$

$$+ (m-1)^2 (n-1)^2 \Big(C_{mn}^2 + C_{mn} C_{m-2,n-2} - C_{mn} C_{m-2,n} - C_{mn} C_{m,n-2} \Big) \Big)$$

$$+ t_i^2 \alpha_{ix} \frac{\pi^3 b}{8a^2} \Big((m+1)^4 \Big(2\gamma B_{mn} A_{mn} - \gamma B_{mn} A_{m+2,n} - B_{mn} A_{m,n-2} + B_{mn} A_{m+3,n-2} - B_{mn} A_{m,n+2}$$

$$+ B_{mn} A_{m+2,n+2} - \gamma A_{mn} B_{m+2,n} - A_{mn} B_{m,n-2} + A_{mn} B_{m+2,n-2} - A_{mn} B_{m,n+2} + A_{mn} B_{m+2,n+2} \Big)$$

$$+ (m-1)^4 \Big(2\gamma B_{mn} A_{mn} - \gamma B_{mn} A_{m-2,n} - B_{mn} A_{m,n+2} + B_{mn} A_{m-2,n+2} + B_{mn} A_{m-2,n-2} - A_{mn} B_{m,n+2}$$

$$- \gamma A_{mn} B_{m-2,n} + A_{mn} B_{m-2,n+2} - A_{mn} B_{m,n-2} - B_{mn} A_{m,n-2} - A_{mn} B_{m-2,n-2} \Big) \Big)$$

$$+t_i^2\,\alpha_{ix}\,\frac{\pi^3}{8b}\Big((m+1)^2(n-1)^2\big(2B_{mn}A_{mn}+B_{mn}A_{m+2,n-2}-B_{mn}A_{m,n-2}-B_{mn}A_{m+2,n}+A_{mn}B_{m+2,n-2}$$

$$-A_{mn}B_{m,n-2}-A_{mn}B_{m+2,n}\big)$$

$$+(m+1)^2(n+1)^2\big(2B_{mn}A_{mn}+B_{mn}A_{m+2,n-2}-B_{mn}A_{m+2,n}-B_{mn}A_{m,n+2}+A_{mn}B_{m+2,n+2}$$

$$-A_{mn}B_{m+2,n}-A_{mn}B_{m,n+2}\big)$$

$$+(m-1)^2(n-1)^2\big(2B_{mn}A_{mn}+B_{mn}A_{m-2,n-2}-B_{mn}A_{m-2,n}-B_{mn}A_{m,n-2}+A_{mn}B_{m-2,n-2}-A_{mn}B_{m-2,n}-A_{mn}B_{m,n-2}\big)$$

$$+(m-1)^2(n+1)^2\big(2B_{mn}A_{mn}+B_{mn}A_{m-2,n+2}-B_{mn}A_{m-2,n}-B_{mn}A_{m,n+2}+A_{mn}B_{m-2,n+2}$$

$$-A_{mn}B_{m-2,n}-A_{mn}B_{m,n+2}\big)\Big)$$

$$+t_i^2\,\alpha_{iy}\,\frac{\pi^3 a}{8b^2}\Big((n+1)^4\big(2\theta C_{mn}A_{mn}-\theta C_{mn}A_{m,n+2}-C_{mn}A_{m-2,n}+C_{mn}A_{m-2,n+2}-C_{mn}A_{m+2,n}$$

$$+C_{mn}A_{m+2,n+2}-\theta A_{mn}C_{m,n+2}+A_{mn}C_{m-2,n+2}-A_{mn}C_{m+2,n}+A_{mn}C_{m+2,n+2}\big)$$

$$+(n-1)^4\big(2\theta C_{mn}A_{mn}-\theta C_{mn}A_{m,n-2}+C_{mn}A_{m-2,n-2}-C_{mn}A_{m-2,n}+C_{mn}A_{m+2,n-2}-C_{mn}A_{m+2,n}$$

$$-\theta A_{mn}C_{m,n-2}+A_{mn}C_{m-2,n-2}-A_{mn}C_{m-2,n}+A_{mn}C_{m+2,n-2}-A_{mn}C_{m+2,n}\big)\Big)$$

$$+t_i^2\,\alpha_{iy}\,\frac{\pi^3}{8a}\Big((m+1)^2(n+1)^2\big(2C_{mn}A_{mn}-C_{mn}A_{m,n+2}-C_{mn}A_{m+2,n}+C_{mn}A_{m+2,n+2}-A_{mn}C_{m,n+2}$$

$$-A_{mn}C_{m+2,n}+A_{mn}C_{m+2,n+2}\big)$$

$$+(m+1)^2(n-1)^2\big(2C_{mn}A_{mn}-C_{mn}A_{m,n-2}-C_{mn}A_{m+2,n}+C_{mn}A_{m+2,n-2}-A_{mn}C_{m,n-2}-A_{mn}C_{m+2,n}+A_{mn}C_{m+2,n-2}\big)$$

$$+ (m-1)^2 (n+1)^2 \Big(2C_{mn}A_{mn} - C_{mn}A_{m-2,n} - C_{mn}A_{m,n+2} + C_{mn}A_{m-2,n+2} - A_{mn}C_{m-2,n}$$

$$- A_{mn}C_{m,n+2} + A_{mn}C_{m-2,n+2} \Big)$$

$$+ (m-1)^2 (n-1)^2 \Big(2C_{mn}A_{mn} \; C_{mn}A_{m-2,n} - C_{mn}A_{m,n-2} + C_{mn}A_{m-2,n-2} - A_{mn}C_{m-2,n}$$

$$- A_{mn}C_{m,n-2} + A_{mn}C_{m-2,n-2} \Big)\Big)$$

$$+ \frac{t_i^3}{3} \frac{\pi^4 b}{4a^3} \Big((m+1)^4 \Big(\gamma A^2_{mn} - A_{mn}A_{m+2,n} - A_{mn}A_{m,n-2} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m,n+2} + A_{mn}A_{m+2,n+2} \Big)$$

$$+ (m-1)^4 \Big(\gamma A^2_{mn} \; \gamma A_{mn}A_{m-2,n} - A_{mn}A_{n+2} + A_{mn}A_{n-2,n+2} - A_{mn}A_{m,n-2} + A_{mn}A_{n-2,n-2} \Big)\Big)$$

$$+ \frac{t_i^3}{3} \frac{\pi^4 a}{4b^3} \Big((n+1)^4 \Big(\theta A^2_{mn} - \theta A_{mn}A_{m,n+2} - A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n+2} \Big)$$

$$+ (n-1)^4 \Big(\theta A^2_{mn} - \theta A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m-2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m+2,n} \Big)\Big)$$

$$+ \frac{2t_i^3}{3} \frac{\pi^4}{4ab} \Big((m+1)^2 (n+1)^2 \Big(A^2_{mn} - A_{mn}A_{m,n+2} - A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n+2} \Big)$$

$$+ (m+1)^2 (n-1)^2 \Big(A^2_{mn} - A_{mn}A_{m,n-2} - A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n-2} \Big)$$

$$+ (m-1)^2 (n+1)^2 \Big(A^2_{mn} - A_{mn}A_{m-2,n} - A_{mn}A_{m,n+2} + A_{mn}A_{m-2,n+2} \Big)$$

$$+ (m-1)^2 (n-1)^2 \Big(A^2_{mn} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m-2,n} - A_{mn}A_{m,n-2} \Big)\Big) \Big\}$$

$$+ \frac{t_c}{2} G_{cxz} \frac{ab}{4} \Big((m+1)^2 \Big(\gamma B^2_{mn} - \gamma B_{mn}B_{m+2,n} - B_{mn}B_{m,n-2} + B_{mn}B_{m+2,n-2} - B_{mn}B_{m,n+2} + B_{mn}B_{m+2,n+2} \Big)$$

$$+ (m-1)^2 \Big(\gamma B^2_{mn} - \gamma B_{mn}B_{m-2,n} + B_{mn}B_{m-2,n-2} - B_{mn}B_{m,n-2} + B_{mn}B_{m-2,n+2} - B_{mn}B_{m,n+2} \Big)\Big)$$

$$+t_c G_{cxz} \frac{\pi b}{8}\Big((m+1)^2\Big(2\gamma B_{mn}A_{mn} - B_{mn}A_{m,n-2} - \gamma B_{mn}A_{m+2,n} + B_{mn}A_{m+2,n-2} - B_{mn}A_{m,n+2} + B_{mn}A_{m+2,n+2}$$

$$- A_{mn}B_{m,n-2} - \gamma A_{mn}B_{n+2,n} + A_{mn}B_{m+2,n-2} - A_{mn}B_{m,n+2} + A_{mn}B_{m+2,n+2}\Big)$$

$$+(m-1)^2\Big(2\gamma B_{mn}A_{mn} - \gamma B_{mn}A_{m-2,n} + B_{mn}A_{m-2,n-2} - B_{mn}A_{m,n-2} + B_{mn}A_{m-2,n+2} - B_{mn}A_{m,n+2}$$

$$-\gamma A_{mn}B_{m-2,n} + A_{mn}B_{m-2,n-2} - A_{mn}B_{m,n-2} + A_{mn}B_{n-2,n+2} - A_{mn}B_{m,n+2}\Big)\Big)$$

$$+\frac{t_c}{2}G_{cxz}\frac{\pi^2 b}{4a}\Big((m+1)^2\Big(\gamma A_{mn}^2 - A_{mn}A_{m,n-2} - \gamma A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m,n+2} + A_{mn}A_{m+2,n+2}\Big)$$

$$+(m-1)^2\Big(\gamma A_{mn}^2 - \gamma A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m,n+2}\Big)\Big)$$

$$+\frac{t_c}{2}G_{cyz}\frac{ab}{4}\Big((n+1)^2\Big(\theta C_{mn}^2 - \theta C_{mn}C_{m,n+2} - C_{mn}C_{m-2,n} + C_{mn}C_{m-2,n+2} - C_{mn}C_{m+2,n} + C_{mn}C_{m+2,n+2}\Big)$$

$$+(n-1)^2\Big(\theta C_{mn}^2 - \theta C_{mn}C_{m,n-2} + C_{mn}C_{m-2,n-2} - C_{mn}C_{m-2,n} + C_{mn}C_{m+2,n-2} - C_{mn}C_{m+2,n}\Big)\Big)$$

$$+t_c G_{cyz}\frac{\pi a}{8}\Big((n+1)^2\Big(2\theta C_{mn}A_{mn} - \theta C_{mn}A_{m,n+2} - C_{mn}A_{m-2,n} + C_{mn}A_{m-2,n+2} - C_{mn}A_{m+2,n} + C_{mn}A_{m+2,n+2}$$

$$-\theta A_{mn}C_{m,n+2} - A_{mn}C_{m-2,n} + A_{mn}C_{m-2,n+2} - A_{mn}C_{m+2,n} + A_{mn}C_{m+2,n+2}\Big)$$

$$+(n-1)^2\Big(\theta C_{mn}A_{mn} - \theta C_{mn}A_{m,n-2} + C_{mn}A_{m-2,n-2} - C_{mn}A_{m-2,n} + C_{mn}A_{m+2,n-2} - C_{mn}A_{m+2,n}$$

$$-\theta A_{mn}C_{m,n-2} + A_{mn}C_{m-2,n-2} - A_{mn}C_{m-2,n} + A_{mn}C_{m+2,n-2} - A_{mn}C_{m+2,n}\Big)\Big)$$

$$+\frac{t_c}{2}G_{cyz}\frac{\pi^2 a}{4b}\Big((n+1)^2\Big(\theta A_{mn}^2 - \theta A_{mn}A_{m,n+2} - A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n+2} + A_{mn}A_{m+2,n+2} - A_{mn}A_{m+2,n}\Big)$$

$$+(n-1)^2\Big(\theta A_{mn}^2 - \theta A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m-2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m+2,n}\Big)\Big)$$

$$+\frac{1}{2}(N_x)\ \frac{\pi^2 b}{4a}\Bigg((m+1)^2\Big(\gamma A_{mn}^2 - A_{mn}A_{m,n-2} - \gamma A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m,n+2} + A_{mn}A_{m+2,n+2}\Big)$$

$$+ (m-1)^2\Big(\gamma A_{mn}^2 - \gamma A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m,n+2}\Big)\Bigg)$$

$$+\frac{1}{2}(N_y)\ \frac{\pi^2 a}{4b}\Bigg((n+1)^2\Big(\theta A_{mn}^2 - \theta A_{mn}A_{m,n+2} - A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n+2}\Big)$$

$$+ (n-1)^2\Big(\theta A_{mn}^2 - \theta A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m-2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m+2,n}\Big)\Bigg)$$

$$+\Bigg\{\sum_s \Delta_{ns}\ N_{xB}\ \frac{b}{2a}\Bigg((m+1)^2\Bigg(A_{mn}A_{ms}\Big(\frac{1}{(n-s)^2} + \frac{1}{(n+s-2)^2} - \frac{1}{(n-s-2)^2} + \frac{1}{(n-s)^2} - \frac{1}{(n+s)^2} - \frac{1}{(n-s+2)^2} - \frac{1}{(n+s)^2} + \frac{1}{(n+s+2)^2}\Big)$$

$$+ A_{mn}A_{m+2,s}\Big(\frac{1}{(n+s)^2} - \frac{1}{(n-s)^2} - \frac{1}{(n+s-2)^2} + \frac{1}{(n-s-2)^2} + \frac{1}{(n+s+2)^2} + \frac{1}{(n+s)^2} - \frac{1}{(n+s)^2} - \frac{1}{(n-s+2)^2}\Big)\Bigg)$$

$$+ (m-1)^2\Bigg(A_{mn}A_{ms}\Big(\frac{1}{(n-s)^2} + \frac{1}{(n+s-2)^2} - \frac{1}{(n-s-2)^2} - \frac{1}{(n+s)^2} - \frac{1}{(n-s+2)^2} - \frac{1}{(n+s)^2} + \frac{1}{(n-s)^2} + \frac{1}{(n+s+2)^2}\Big)$$

$$+ A_{mn}A_{m-s,s}\Big(\frac{1}{(n+s)^2} - \frac{1}{(n-s)^2} - \frac{1}{(n+s-2)^2} + \frac{1}{(n-s-2)^2} + \frac{1}{(n-s+2)^2} + \frac{1}{(n+s)^2} - \frac{1}{(n-s)^2} - \frac{1}{(n+s+2)^2}\Big)\Bigg)\Bigg)$$

$$+\sum_r \Delta_{mr}\ N_{yB}\ \frac{a}{2b}\Bigg((n+1)^2\Bigg(A_{mn}A_{rn}\Big(\frac{1}{(m-r)^2} + \frac{1}{(m+r-2)^2} - \frac{1}{(m-r-2)^2} - \frac{1}{(m+r)^2} - \frac{1}{(m-r+2)} - \frac{1}{(m+r)^2}$$

$$+ \frac{1}{(m-r)^2} + \frac{1}{(m+r+2)^2}\Big) + A_{mn}A_{r,n+2}\Big(\frac{1}{(m+r)^2} - \frac{1}{(m-r)^2} - \frac{1}{(m+r-2)^2} + \frac{1}{(m-r-2)^2}$$

$$+ \frac{1}{(m-r+2)^2} + \frac{1}{(m+r)^2} - \frac{1}{(m-r)^2} - \frac{1}{(m+r+2)^2}\Big)\Bigg)$$

$$+ (n-1)^2\Bigg(A_{mn}A_{rn}\Big(\frac{1}{(m-r)^2} + \frac{1}{(m+r-2)^2} - \frac{1}{(m-s-2)^2} - \frac{1}{(m+r)^2} - \frac{1}{(m-r+2)} - \frac{1}{(m+r)^2}$$

$$+ \frac{1}{(m-r)^2} + \frac{1}{(m+r+2)^2}\Big) + A_{mn}A_{r,n-2}\Big(\frac{1}{(m+r)^2} - \frac{1}{(m-r)^2} - \frac{1}{(m+r-2)^2} + \frac{1}{(m-r-2)^2}$$

$$+ \frac{1}{(m-r+2)^2} + \frac{1}{(m+r)^2} - \frac{1}{(m-r)^2} - \frac{1}{(m+r+2)^2}\Big)\Bigg)\Bigg)$$

$$
+\sum_{r}\sum_{s}\Delta_{mr}\Delta_{ns}\,N_{xy}\;2A_{mn}\,A_{rs}\left(\frac{(m+1)^2(s+1)^2+(r-1)^2(n-1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s+1)^2-(n-1)^2\right)}-\frac{(m+1)^2(s-1)^2+(r-1)^2(n-1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s-1)^2-(n-1)^2\right)}\right.
$$

$$
\frac{(m+1)^2(s+1)^2+(r+1)^2(n-1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s+1)^2-(n-1)^2\right)}+\frac{(m+1)^2(s-1)^2+(r+1)^2(n-1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s-1)^2-(n-1)^2\right)}-\frac{(m+1)^2(s+1)^2+(r-1)^2(n+1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s+1)^2-(n+1)^2\right)}
$$

$$
+\frac{(m+1)^2(s-1)^2+(r-1)^2(n+1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s-1)^2-(n-1)^2\right)}+\frac{(m+1)^2(s+1)^2+(r+1)^2(n+1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s+1)^2-(n+1)^2\right)}-\frac{(m+1)^2(s-1)^2+(r+1)^2(n+1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)}
$$

$$
-\frac{(m-1)^2(s+1)^2+(r-1)^2(n-1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s+1)^2-(n-1)^2\right)}+\frac{(m-1)^2(s-1)^2+(r-1)^2(n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)}+\frac{(m-1)^2(s+1)^2+(r+1)^2(n-1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s+1)^2-(n-1)^2\right)}
$$

$$
-\frac{(m-1)^2(s-1)^2+(r+1)^2(n-1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n-1)^2\right)}+\frac{(m-1)^2(s+1)^2+(r-1)^2(n+1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s-1)^2-(n+1)^2\right)}-\frac{(m-1)^2(s-1)^2+(r-1)^2(n+1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s-1)^2-(n+1)^2\right)}
$$

$$
\left.-\frac{(m-1)^2(s+1)^2+(r+1)^2(n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s+1)^2-(n+1)^2\right)}+\frac{(m-1)^2(s-1)^2+(r+1)^2(n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)}\right)\Big\}\Big]
$$

APPENDIX C

POTENTIAL ENERGY IN QUADRATIC FORM

APPENDIX C

POTENTIAL ENERGY IN QUADRATIC FORM

This Appendix presents the final expression for the potential energy of the sandwich panel. This includes Equation (8) into the Equation (2) and Appendix B where

$$\gamma = \begin{cases} 2, n \neq 1 \\ 3, n = 1 \end{cases} \quad \text{and} \quad \theta = \begin{cases} 2, n \neq 1 \\ 3, n = 1 \end{cases}$$

$$\pi_p = \sum_m \sum_n \Bigg[\sum_{i=1}^{2} \frac{1}{2} \frac{E_i}{\lambda_i} \Bigg\{ t_i \beta_i^2 \frac{\pi^2 b}{4a} \Bigg((m+1)^4 \Big(\gamma B_{mn}^2 - \gamma B_{mn} B_{m+2,n} - B_{mn} B_{m,n-2} + B_{mn} B_{m+2,n-2}$$

$$- B_{mn} B_{m,n+2} + B_{mn} B_{m+2,n+2} \Big) \Bigg|$$

$$+ (m-1)^4 \Big(\gamma B_{mn}^2 - \gamma B_{mn} B_{m-2,n} - B_{mn} B_{m,n+2} + B_{mn} B_{m-2,n+2} - B_{mn} B_{m,n-2} + B_{mn} B_{m-2,n-2} \Big) \Bigg)$$

$$+ t_i (-1)^i \beta_i \frac{\pi^2 b}{4a} \Bigg((m+1)^4 \Big(2\gamma B_{mn} H_{mn} - \gamma H_{mn} B_{m+2,n} - H_{mn} B_{m,n-2} + H_{mn} B_{m+2,n-2} - H_{mn} B_{m,n+2}$$

$$+ H_{mn} B_{m+2,n+2} + B_{mn} H_{m+2,n-2} - \gamma B_{mn} H_{m+2,n} - B_{mn} H_{m,n-2} - B_{mn} H_{m,n+2} + B_{mn} H_{m+2,n+2} \Big)$$

$$+ (m-1)^4 \Big(2\gamma B_{mn} H_{mn} - \gamma B_{mn} H_{m-2,n} + B_{mn} H_{m-2,n+2} - \gamma B_{mn} H_{m-2,n} - B_{mn} H_{m,n+2} - B_{mn} H_{m,n-2}$$

$$- H_{mn} B_{m,n+2} + H_{mn} B_{m-2,n+2} - H_{mn} B_{m,n-2} + H_{mn} B_{m-2,n-2} - B_{mn} H_{m-2,n-2} \Big) \Bigg)$$

$$+ t_i \frac{\pi^2 b}{4a} \Bigg((m+1)^4 \Big(\gamma H_{mn}^2 - \gamma H_{mn} H_{m+2,n} - H_{mn} H_{m,n-2} + H_{mn} H_{m+2,n-2} - H_{mn} H_{m,n+2} + H_{mn} H_{m+2,n+2} \Big)$$

$$+ (m-1)^4 \Big(\gamma H_{mn}^2 - \gamma H_{mn} H_{m-2,n} - H_{mn} H_{m,n+2} + H_{mn} H_{m-2,n+2} - H_{mn} H_{m,n-2} + H_{mn} H_{m-2,n-2} \Big) \Bigg)$$

$$+ t_i \left(\frac{1-\nu_i}{2}\right) \beta_i^2 \; \frac{\pi^2 a}{4b} \Big((m+1)^2 (n+1)^2 \Big(B_{mn}^2 - B_{mn}B_{m,n+2} - B_{mn}B_{m+2,n} + B_{mn}B_{m+2,n+2} \Big)$$

$$+(m+1)^2 (n-1)^2 \Big(B_{mn}^2 - B_{mn}B_{m,n-2} - B_{mn}B_{m+2,n} + B_{mn}B_{m+2,n-2} \Big)$$

$$+(m-1)^2 (n+1)^2 \Big(B_{mn}^2 - B_{mn}B_{m-2,n} - B_{mn}B_{m,n+2} + B_{mn}B_{m-2,n+2} \Big)$$

$$+(m-1)^2 (n-1)^2 \Big(B_{mn}^2 + B_{mn}B_{m-2,n-2} - B_{mn}B_{m-2,n} - B_{mn}B_{m,n-2} \Big)\Big)$$

$$+ t_i \left(\frac{1-\nu_i}{2}\right) (-1)^2 \; \beta_i \; \frac{\pi^2 a}{4b} \Big((m+1)^2 (n+1)^2 \Big(2H_{mn}B_{mn} - H_{mn}B_{m,n+2} - H_{mn}B_{m+2,n} + H_{mn}B_{m+2,n+2}$$

$$- B_{mn}H_{m,n+2} - B_{mn}H_{m+2,n} + B_{mn}H_{m+2,n+2} \Big)$$

$$+(m+1)^2 (n-1)^2 \Big(2H_{mn}B_{mn} + H_{mn}B_{m+2,n-2} - H_{mn}B_{m+2,n} - H_{mn}B_{m,n-2} + B_{mn}H_{m+2,n-2} - B_{mn}H_{m+2,n}$$

$$- B_{mn}H_{m,n-2} \Big)$$

$$+(m-1)^2 (n+1)^2 \Big(2B_{mn}H_{mn} - H_{mn}B_{m-2,n} - H_{mn}B_{m,n+2} + H_{mn}B_{m-2,n+2} - B_{mn}H_{m-2,n}$$

$$- B_{mn}H_{m,n+2} + B_{mn}H_{m-2,n+2} \Big)$$

$$+(m-1)^2 (n-1)^2 \Big(2H_{mn}B_{mn} + H_{mn}B_{m-2,n-2} - H_{mn}B_{m-2,n} - H_{mn}B_{m,n-2} + B_{mn}H_{m-2,n-2}$$

$$- B_{mn}H_{m-2,n} - B_{mn}H_{m,n-2} \Big)\Big)$$

$$+ t_i \left(\frac{1-\nu_i}{2}\right) \frac{\pi^2 a}{4b} \Big((m+1)^2 (n+1)^2 \Big(H_{mn}^2 - H_{mn}H_{m,n+2} - H_{mn}H_{m+2,n} + H_{mn}H_{m+2,n+2} \Big)$$

$$+(m+1)^2 (n-1)^2 \Big(H_{mn}^2 + H_{mn}H_{m+2,n-2} - H_{mn}H_{m+2,n} - H_{mn}H_{m,n-2} \Big)$$

$$+(m-1)^2\,(n+1)^2\Big(H^2_{mn} - H_{mn}H_{m-2,n} - H_{mn}H_{m,n+2} + H_{mn}H_{m-2,n+2}\Big)$$

$$+(m-1)^2\,(n-1)^2\Big(H^2_{mn} + H_{mn}H_{m-2,n-2} - H_{mn}H_{m-2,n} - H_{mn}H_{m,n-2}\Big)\Big)$$

$$+t_i\,(1+\nu_i)\,\beta_i^2\,\frac{\pi^2}{8}\Big((m+1)^2\,(n-1)^2\Big(2B_{mn}C_{mn} + B_{mn}C_{m+2,\,n-2} - B_{mn}C_{m,n-2} - B_{mn}C_{m+2,n}$$

$$+C_{mn}B_{m+2,n-2} - C_{mn}B_{m,n-2} - C_{mn}B_{m+2,\,n}\Big)$$

$$+(m+1)^2\,(n+1)^2\Big(2B_{mn}C_{mn} + B_{mn}C_{m+2,\,n+2} - B_{mn}C_{m+2,n} - B_{mn}C_{m,n+2} + C_{mn}B_{m+2,\,n+2}$$

$$-C_{mn}B_{m+2,\,n} - C_{mn}B_{m,n+2}\Big)$$

$$+(m-1)^2\,(n-1)^2\Big(2B_{mn}C_{mn} + B_{mn}C_{m-2,n-2} - B_{mn}C_{m,n-2} - B_{mn}C_{m-2,n} + C_{mn}B_{m-2,n-2}$$

$$-C_{mn}B_{m-2,n} - C_{mn}B_{m,n-2}\Big)$$

$$+(m-1)^2\,(n+1)^2\Big(2B_{mn}C_{mn} + B_{mn}C_{m-2,n+2} - B_{mn}C_{m-2,n} - B_{mn}C_{m,n+2} + C_{mn}B_{m-2,\,n+2}$$

$$-C_{mn}B_{m,n+2}\Big)\Big)$$

$$+t_i\,(1+\nu_i)\,(-1)^i\,\beta_i\,\frac{\pi^2}{8}\Big((m+1)^2\,(n-1)^2\Big(2H_{mn}C_{mn} + H_{mn}C_{m+2,n-2} - H_{mn}C_{m,n-2} - H_{mn}C_{m+2,n} + C_{mn}H_{m+2,n-2}$$

$$-C_{mn}H_{m,n-2} - C_{mn}H_{m+2,n} + 2B_{mn}K_{mn} + B_{mn}K_{m+2,n-2} - B_{mn}K_{m,n-2}$$

$$-B_{mn}K_{m+2,n} + K_{mn}B_{m+2,n-2} - K_{mn}B_{m,n-2} - K_{mn}B_{m+2,n}\Big)$$

$$+(m+1)^2\,(n+1)^2\Big(2H_{mn}C_{mn} + H_{mn}C_{m+2,n+2} - H_{mn}C_{m+2,n} - H_{mn}C_{m,n+2} + C_{mn}H_{m+2,n+2} - C_{mn}H_{m+2,n}$$

$$+B_{mn}K_{m+2,n+2} - B_{mn}K_{m+2,n} - B_{mn}K_{m,n+2} + K_{mn}B_{m+2,n+2} - C_{mn}H_{m,n+2} + 2B_{mn}K_{mn}$$

$$- K_{mn}B_{m+2,n} - K_{mn}B_{m,n+2}\Big)$$

$$+(m-1)^2(n-1)^2\Big(2H_{mn}C_{mn} + H_{mn}C_{m-2,n-2} - H_{mn}C_{m,n-2} - H_{mn}C_{m-2,n} + C_{mn}H_{m-2,n-2}$$

$$- C_{mn}H_{m-2,n} - C_{mn}H_{m,n-2} + 2B_{mn}K_{mn} + B_{mn}K_{m-2,n-2} - B_{mn}K_{m,n-2}$$

$$- B_{mn}K_{m-2,n} + K_{mn}B_{m-2,n-2} - K_{mn}B_{m-2,n} - K_{mn}B_{m,n-2}\Big)$$

$$+(m-1)^2(n+1)^2\Big(2H_{mn}C_{mn} + H_{mn}C_{m-2,n+2} - H_{mn}C_{m-2,n} - H_{mn}C_{m,n+2} + C_{mn}H_{m-2,n+2} - C_{mn}H_{m-2,n}$$

$$- C_{mn}H_{m,n+2} + 2B_{mn}K_{mn} + B_{mn}K_{m-2,n+2} - B_{mn}K_{m-2,n} - B_{mn}K_{m,n+2}$$

$$+K_{mn}B_{m-2,n+2} - K_{mn}B_{m-2,n} - K_{mn}B_{m,n+2}\Big)\Big)$$

$$+t_i(1+\nu_i)\frac{\pi^2}{8}\Big((m+1)^2(n-1)^2\Big(2H_{mn}K_{mn} + H_{mn}K_{m+2,n-2} - H_{mn}K_{m,n-2} - H_{mn}K_{m+2,n} + K_{mn}H_{m+2,n-2}$$

$$- K_{mn}H_{m,n-2} - K_{mn}H_{m+2,n}\Big)$$

$$+(m+1)^2(n+1)^2\Big(2H_{mn}K_{mn} + H_{mn}K_{m+2,n+2} - H_{mn}K_{m+2,n} - H_{mn}K_{m,n+2} + K_{mn}H_{m+2,n+2}$$

$$- K_{mn}H_{m+2,n} - K_{mn}H_{m,n+2}\Big)$$

$$+(m-1)^2(n-1)^2\Big(2H_{mn}K_{mn} + H_{mn}K_{m-2,n-2} - H_{mn}K_{m,n-2} - H_{mn}K_{m-2,n} + K_{mn}H_{m-2,n-2}$$

$$- K_{mn}H_{m-2,n} - K_{mn}H_{m,n-2}\Big)$$

$$+(m-1)^2\,(n+1)^2\Big(2H_{mn}K_{mn} + H_{mn}K_{m-2,\,n+2} - H_{mn}K_{m-2,\,n} - H_{mn}K_{m,\,n+2} + K_{mn}H_{m-2,\,n+2}$$

$$- K_{mn}H_{m-2,\,n} - K_{mn}H_{m,\,n+2}\Big)\Big)$$

$$+t_i\,\beta_i^2\,\frac{\pi^2 a}{4b}\Big((n+1)^4\Big(9C_{mn}^2 - 9C_{mn}C_{m,n+2} - C_{mn}C_{m-2,n} + C_{mn}C_{m-2,n+2} - C_{mn}C_{m+2,n} + C_{mn}C_{m+2,\,n+2}\Big)$$

$$+(n-1)^4\Big(9\,C_{mn}^2 - 9C_{mn}C_{m,n-2} + C_{mn}C_{m-2,\,n-2} - C_{mn}C_{m-2,\,n} + C_{mn}C_{m+2,\,n-2} - C_{mn}C_{m+2,\,n}\Big)\Big)$$

$$+t_i\,\beta_i\,(-1)^i\,\frac{\pi^2 a}{4b}\Big((n+1)^4\Big(2\,9\,C_{mn}K_{mn} - 9\,K_{mn}C_{m,n+2} - K_{mn}C_{m-2,n} + K_{mn}C_{m-2,n+2} - K_{mn}C_{m+2,\,n}$$

$$+K_{mn}C_{m+2,\,n+2} - 9\,C_{mn}K_{m,n+2} - C_{mn}K_{m-2,n} + C_{mn}K_{m-2,n+2} - C_{mn}K_{m+2,n} + C_{mn}K_{m+2,\,n+2}\Big)$$

$$+(n-1)^4\Big(29\,C_{mn}K_{mn} - 9\,K_{mn}C_{m,n-2} + K_{mn}C_{m-2,n-2} - K_{mn}C_{m-2,n} + K_{mn}C_{m+2,n-2} - K_{mn}C_{m+2,n}$$

$$- 9\,C_{mn}K_{m,n-2} + C_{mn}K_{m-2,n-2} - C_{mn}K_{m-2,n} + C_{mn}K_{m+2,n-2} - C_{mn}K_{m+2,n}\Big)\Big)$$

$$+t_i\,\frac{\pi^2 a}{4b}\Big((n+1)^4\Big(9\,K_{mn}^2 - 9\,K_{mn}K_{m,n+2} - K_{mn}K_{m-2,n} + K_{mn}K_{m-2,n+2} - K_{mn}K_{m+2,n} + K_{mn}K_{m+2,\,n+2}\Big)$$

$$+(n-1)^4\Big(9\,K_{mn}^2 - 9K_{mn}K_{m,n-2} + K_{mn}K_{m-2,n-2} - K_{mn}K_{m-2,n} + K_{mn}K_{m+2,\,n-2} - K_{mn}K_{m+2,\,n}\Big)\Big)$$

$$+t_i\left(\frac{1-\nu_i}{2}\right)\beta_i^2\,\frac{\pi^2 b}{4a}\Big((m+1)^2\,(n+1)^2\Big(C_{mn}^2 - C_{mn}C_{m,n+2} - C_{mn}C_{m+2,n} + C_{mn}C_{m+2,n+2}\Big)$$

$$+(m+1)^2\,(n-1)^2\Big(C_{mn}^2 - C_{mn}C_{m,n-2} - C_{mn}C_{m+2,\,n} + C_{mn}C_{m+2,\,n-2}\Big)$$

$$+(m-1)^2\,(n+1)^2\Big(C_{mn}^2 - C_{mn}C_{m-2,n} - C_{mn}C_{m,n+2} + C_{mn}C_{m-2,n+2}\Big)$$

$$+(m-1)^2\,(n-1)^2\Big(C_{mn}^2 + C_{mn}C_{m-2,n-2} - C_{mn}C_{m-2,n} - C_{mn}C_{m,n-2}\Big)\Big)$$

$$+t_i\left(\frac{1-\nu_i}{2}\right)(-1)^i\,\beta_i\,\frac{\pi^2 b}{4a}\Big((m+1)^2(n+1)^2\big(2C_{mn}K_{mn} - C_{mn}K_{m,n+2} - C_{mn}K_{m+2,n} + C_{mn}K_{m+2,n+2} - K_{mn}C_{m,n+2}$$

$$- K_{mn}C_{m+2,n} + K_{mn}C_{m+2,n+2}\big)$$

$$+(m+1)^2(n-1)^2\big(2C_{mn}K_{mn} - C_{mn}K_{m,n-2} - C_{mn}K_{m+2,n} + C_{mn}K_{m+2,n-2} - K_{mn}C_{m,n-2} - K_{mn}C_{m+2,n}$$

$$+K_{mn}C_{m+2,n-2}\big)$$

$$+(m-1)^2(n+1)^2\big(2C_{mn}K_{mn} - C_{mn}K_{m-2,n} - C_{mn}K_{m,n+2} + C_{mn}K_{m-2,n+2} - K_{mn}C_{m-2,n} - K_{mn}C_{m,n+2}$$

$$+K_{mn}C_{m-2,n+2}\big)$$

$$+(m-1)^2(n-1)^2\big(2C_{mn}K_{mn} - C_{mn}K_{m-2,n} - C_{mn}K_{m,n-2} + C_{mn}K_{m-2,n-2} - K_{mn}C_{m-2,n} - K_{mn}C_{m,n-2}$$

$$+K_{mn}C_{m-2,n-2}\big)\Big)$$

$$+t_i\left(\frac{1-\nu_i}{2}\right)\frac{\pi^2 b}{4a}\Big((m+1)^2(n+1)^2\big(K_{mn}^2 - K_{mn}K_{m,n+2} - K_{mn}K_{m+2,n} + K_{mn}K_{m+2,n+2}\big)$$

$$+(m+1)^2(n-1)^2\big(K_{mn}^2 - K_{mn}K_{m,n-2} - K_{mn}K_{m+2,n} + K_{mn}K_{m+2,n-2}\big)$$

$$+(m-1)^2(n+1)^2\big(K_{mn}^2 - K_{mn}K_{m,n+2} - K_{mn}K_{m-2,n} + K_{mn}K_{m-2,n+2}\big)$$

$$+(m-1)^2(n-1)^2\big(K_{mn}^2 - K_{mn}K_{m,n-2} - K_{mn}K_{m-2,n} + K_{mn}K_{m-2,n-2}\big)\Big)$$

$$+t_i^2\,\beta_i\,\frac{\pi^3 b}{8a^2}\Big((m+1)^2\big(2\gamma B_{mn}A_{mn} - \gamma B_{mn}A_{m+2,n} - B_{mn}A_{m,n-2} + B_{mn}A_{m+2,n-2} - B_{mn}A_{m,n+2} + B_{mn}A_{m+2,n+2}$$

$$-\gamma A_{mn}B_{m+2,n} - A_{mn}B_{m,n-2} + A_{mn}B_{m+2,n-2} - A_{mn}B_{m,n+2} + A_{mn}B_{m+2,n+2}\big)$$

$$+(m-1)^4\Big(2\gamma\, B_{mn}A_{mn} - \gamma\, B_{mn}A_{m-2,n} - B_{mn}A_{m,n+2} + B_{mn}A_{m-2,n+2} - B_{mn}A_{m,n-2} + B_{mn}A_{m-2,n-2}$$

$$-\gamma A_{mn}B_{m-2,n} - A_{mn}B_{m,n+2} + A_{mn}B_{m-2,n+2} - A_{mn}B_{m,n-2} + A_{mn}B_{m-2,n-2}\Big)\Big)$$

$$+t_i^2\,(-1)^i\;\frac{\pi^3 b}{8a^2}\Big((m+1)^4\Big(2\gamma\, H_{mn}A_{mn} - \gamma\, H_{mn}A_{m+2,n} - H_{mn}A_{m,n-2} + H_{mn}A_{m+2,n-2} - H_{mn}A_{m,n+2} + H_{mn}A_{m+2,n+2}$$

$$-\gamma A_{mn}H_{m+2,n} - A_{mn}H_{m,n-2} + A_{mn}H_{m+2,n-2} - A_{mn}H_{m,n+2} + A_{mn}H_{m+2,n+2}\Big)$$

$$+(m-1)^4\Big(2\gamma\, H_{mn}A_{mn} - \gamma\, H_{mn}A_{m-2,n} - H_{mn}A_{m,n+2} + H_{mn}A_{m-2,n+2} - H_{mn}A_{m,n-2} + H_{mn}A_{m-2,n-2}$$

$$-\gamma A_{mn}H_{m-2,n} - A_{mn}H_{m,n+2} + A_{mn}H_{m-2,n+2} + A_{mn}H_{m-2,n-2} - A_{mn}H_{m,n-2}\Big)\Big)$$

$$+t_i^2\,\beta_i\;\frac{\pi^3}{8b}\Big((m+1)^2(n-1)^2\Big(2B_{mn}A_{mn} + B_{mn}A_{m+2,n-2} - B_{mn}A_{m,n-2} - B_{mn}A_{m+2,n} + A_{mn}B_{m+2,n-2}$$

$$- A_{mn}B_{m,n-2} - A_{mn}B_{m+2,n}\Big)$$

$$+(m+1)^2(n+1)^2\Big(2B_{mn}A_{mn} + B_{mn}A_{m+2,n+2} - B_{mn}A_{m+2,n} - B_{mn}A_{m,n+2} + A_{mn}B_{m+2,n+2}$$

$$- A_{mn}B_{m+2,n} - A_{mn}B_{m,n+2}\Big)$$

$$+(m-1)^2(n-1)^2\Big(2B_{mn}A_{mn} + B_{mn}A_{m-2,n-2} - B_{mn}A_{m-2,n} - B_{mn}A_{m,n-2} + A_{mn}B_{m-2,n-2}$$

$$- A_{mn}B_{m-2,n} - A_{mn}B_{m,n-2}\Big)$$

$$+(m-1)^2(n+1)^2\Big(2B_{mn}A_{mn} + B_{mn}A_{m-2,n+2} - B_{mn}A_{m-2,n} - B_{mn}A_{m,n+2} + A_{mn}B_{m-2,n+2}$$

$$- A_{mn}B_{m-2,n} - A_{mn}B_{m,n+2}\Big)\Big)$$

$$+t_i^2\ (-1)^i\ \frac{\pi^3}{8b}\Big((m+1)^2\ (n-1)^2\Big(2H_{mn}A_{mn}+H_{mn}A_{m+2,n-2}-H_{mn}A_{m,n-2}-H_{mn}A_{m+2,n}+A_{mn}H_{m+2,n-2}$$

$$-A_{mn}H_{m,n-2}-A_{mn}H_{m+2,n}\Big)$$

$$+(m+1)^2\ (n+1)^2\Big(2H_{mn}A_{mn}+H_{mn}A_{m+2,n+2}-H_{mn}A_{m+2,n}-H_{mn}A_{m,n+2}+A_{mn}H_{m+2,n+2}$$

$$-A_{mn}H_{m+2,n}-A_{mn}H_{m,n+2}\Big)$$

$$+(m-1)^2\ (n-1)^2\Big(2H_{mn}A_{mn}+H_{mn}A_{m-2,n-2}-H_{mn}A_{m-2,n}-H_{mn}A_{m,n-2}+A_{mn}H_{m-2,n-2}$$

$$-A_{mn}H_{m-2,n}-A_{mn}H_{m,n-2}\Big)$$

$$+(m-1)^2\ (n+1)^2\Big(2H_{mn}A_{mn}+H_{mn}A_{m-2,n+2}-H_{mn}A_{m-2,n}-H_{mn}A_{m,n+2}+A_{mn}H_{m-2,n+2}$$

$$-A_{mn}H_{m-2,n}-A_{mn}H_{m,n+2}\Big)\Big)$$

$$+t_i^2\ \beta_i\ \frac{\pi^3 a}{8b^2}\Big((n+1)^4\Big(2\theta C_{mn}A_{mn}-\theta\ C_{mn}A_{m,n+2}-C_{mn}A_{m-2,n}+C_{mn}A_{m-2,n+2}-C_{mn}A_{m+2,n}$$

$$+C_{mn}A_{m+2,n+2}-\theta\ A_{mn}C_{m,n+2}\ A_{mn}C_{m-2,n}+A_{mn}C_{m-2,n+2}-A_{mn}C_{m+2,n}+A_{mn}C_{m+2,n+2}\Big)$$

$$+(n-1)^4\Big(2\theta C_{mn}A_{mn}-\theta C_{mn}A_{m,n-2}+C_{mn}A_{m-2,n-2}-C_{mn}A_{m-2,n}+C_{mn}A_{m+2,n-2}-C_{mn}A_{m+2,n}$$

$$-\theta\ A_{mn}C_{m,n-2}+A_{mn}C_{m-2,n-2}-A_{mn}C_{m-2,n}+A_{mn}C_{m+2,n-2}-A_{mn}C_{m+2,n}\Big)\Big)$$

$$+t_i^2\ (-1)^i\ \frac{\pi^3 a}{8b^2}\Big((n+1)^4\Big(2\theta\ K_{mn}A_{mn}-\theta\ K_{mn}A_{m,n+2}-K_{mn}A_{m-2,n}+K_{mn}A_{m-2,n+2}-K_{mn}A_{m+2,n}+K_{mn}A_{m+2,n+2}$$

$$-\theta\ A_{mn}K_{m,n+2}-A_{mn}K_{m-2,n}+A_{mn}K_{m-2,n+2}-A_{mn}K_{m+2,n}+A_{mn}K_{m+2,n+2}\Big)$$

$$+(n-1)^4\Big(2\theta K_{mn}A_{mn} - \theta K_{mn}A_{m,n-2} + K_{mn}A_{m-2,n-2} - K_{mn}A_{m-2,n} + K_{mn}A_{m+2,n-2}$$

$$- K_{mn}A_{m+2,n} - \theta A_{mn}K_{m,n-2} + A_{mn}K_{m-2,n-2} - A_{mn}K_{m-2,n} + A_{mn}K_{m+2,n-2} - A_{mn}K_{m+2,n})\Big)$$

$$+ t_i^2 \beta_i \frac{\pi^3}{8a}\Big((m+1)^2(n+1)^2\big(2C_{mn}A_{mn} - C_{mn}A_{m,n+2} - C_{mn}A_{m+2,n} + C_{mn}A_{m+2,n+2} - A_{mn}C_{m,n+2}$$

$$- A_{mn}C_{m+2,n} + A_{mn}C_{m+2,n+2}\big)$$

$$+(m+1)^2(n-1)^2\big(2C_{mn}A_{mn} - C_{mn}A_{m,n-2} - C_{mn}A_{m+2,n} + C_{mn}A_{m+2,n-2} - A_{mn}C_{m,n-2}$$

$$- A_{mn}C_{m+2,n} + A_{mn}C_{m+2,n-2}\big)$$

$$+(m-1)^2(n+1)^2\big(2C_{mn}A_{mn} - C_{mn}A_{m-2,n} - C_{mn}A_{m,n+2} + C_{mn}A_{m-2,n+2} - A_{mn}C_{m-2,n}$$

$$- A_{mn}C_{m,n+2} + A_{mn}C_{m-2,n+2}\big)$$

$$+(m-1)^2(n-1)^2\big(2C_{mn}A_{mn} - C_{mn}A_{m-2,n} - C_{mn}A_{m,n-2} + C_{mn}A_{m-2,n-2} - A_{mn}C_{m-2,n}$$

$$- A_{mn}C_{m,n-2} + A_{mn}C_{m-2,n-2}\big)\Big)$$

$$+t_i^2(-1)^i\frac{\pi^3}{8a}\Big((m+1)^2(n+1)^2\big(2K_{mn}A_{mn} - K_{mn}A_{m,n+2} - K_{mn}A_{m+2,n}\ K_{mn}A_{m+2,n+2} - A_{mn}K_{m,n+2}$$

$$- A_{mn}K_{m+2,n} + A_{mn}K_{m+2,n+2}\big)$$

$$+(m+1)^2(n-1)^2\big(2K_{mn}A_{mn} - K_{mn}A_{m,n-2} - K_{mn}A_{m+2,n} + K_{mn}A_{m+2,n-2} - A_{mn}K_{m,n-2}$$

$$- A_{mn}K_{m+2,n} + A_{mn}K_{m+2,n-2}\big)$$

$$+(m-1)^2 (n+1)^2 \Big(2K_{mn} A_{mn} - K_{mn} A_{m-2,n} - K_{mn} A_{m,n+2} + K_{mn} A_{m-2,n+2} - A_{mn} K_{m-2,n}$$

$$- A_{mn} K_{m,n+2} + A_{mn} K_{m-2,n+2} \Big)$$

$$+(m-1)^2 (n-1)^2 \Big(2K_{mn} A_{mn} - K_{mn} A_{m-2,n} - K_{mn} A_{m,n-2} + K_{mn} A_{m-2,n-2} - A_{mn} K_{m-2,n}$$

$$- A_{mn} K_{m,n-2} + A_{mn} K_{m-2,n-2} \Big)\Big)$$

$$+ \frac{t_i^3}{3} \frac{\pi^4 b}{4a^3} \Big((m+1)^4 \Big(\gamma A^2_{mn} - A_{mn} A_{m+2,n} - A_{mn} A_{m,n-2} + A_{mn} A_{m+2,n-2} - A_{mn} A_{m,n+2} + A_{mn} A_{m+2,n+2} \Big)$$

$$+(m-1)^4 \Big(\gamma A^2_{mn} \; \gamma A_{mn} A_{m-2,n} - A_{mn} A_{n+2} + A_{mn} A_{n-2,n+2} - A_{mn} A_{m,n-2} + A_{mn} A_{n-2,n-2} \Big)\Big)$$

$$+ \frac{t_i^3}{3} \frac{\pi^4 a}{4b^3} \Big((n+1)^4 \Big(\theta A^2_{mn} - \theta A_{mn} A_{m,n+2} - A_{mn} A_{m-2,n} + A_{mn} A_{m-2,n+2} - A_{mn} A_{m+2,n} + A_{mn} A_{m+2,n+2} \Big)$$

$$+ (n-1)^4 \Big(\theta A^2_{mn} - \theta A_{mn} A_{m,n-2} + A_{mn} A_{m-2,n-2} - A_{mn} A_{m-2,n} + A_{mn} A_{m+2,n-2} - A_{mn} A_{m+2,n} \Big)\Big)$$

$$+ \frac{2t_i^3}{3} \frac{\pi^4}{4ab} \Big((m+1)^2 (n+1)^2 \Big(A^2_{mn} - A_{mn} A_{m,n+2} - A_{mn} A_{m+2,n} + A_{mn} A_{m+2,n+2} \Big)$$

$$+(m+1)^2 (n-1)^2 \Big(A^2_{mn} - A_{mn} A_{m,n-2} - A_{mn} A_{m+2,n} + A_{mn} A_{m+2,n-2} \Big)$$

$$+(m-1)^2 (n+1)^2 \Big(A^2_{mn} - A_{mn} A_{m-2,n} - A_{mn} A_{m,n+2} + A_{mn} A_{m-2,n+3} \Big)$$

$$+(m-1)^2 (n-1)^2 \Big(A^2_{mn} + A_{mn} A_{m-2,n-2} - A_{mn} A_{m-2,n} \cdot A_{mn} A_{m,n-2} \Big)\Big) \Big\}$$

$$+ \frac{t_c}{2} G_{cxz} \frac{ab}{4} \Big((m+1)^2 \Big(\gamma B^2_{mn} - \gamma B_{mn} B_{m+2,n} - B_{mn} B_{m,n-2} + B_{mn} B_{m+2,n-2} - B_{mn} B_{m,n+2} + B_{mn} B_{m+2,n+2} \Big)$$

$$+ (m-1)^2 \Big(\gamma B^2_{mn} - \gamma B_{mn} B_{m-2,n} + B_{mn} B_{m-2,n-2} - B_{mn} B_{m,n-2} + B_{mn} B_{m-2,n+2} - B_{mn} B_{m,n+2} \Big)\Big)$$

$$+t_c G_{cxz} \frac{\pi b}{8}\Big((m+1)^2\Big(2\gamma B_{mn} A_{mn} - B_{mn} A_{m,n-2} - \gamma B_{mn} A_{m+2,n} + B_{mn} A_{m+2,n-2} - B_{mn} A_{m,n+2} + B_{mn} A_{m+2,n+2}$$

$$- A_{mn} B_{m,n-2} - \gamma A_{mn} B_{n+2,n} + A_{mn} B_{m+2,n-2} - A_{mn} B_{m,n+2} + A_{mn} B_{m+2,n+2}\Big)$$

$$+(m-1)^2\Big(2\gamma B_{mn} A_{mn} - \gamma B_{mn} A_{m-2,n} + B_{mn} A_{m-2,n-2} - B_{mn} A_{m,n-2} + B_{mn} A_{m-2,n+2} - B_{mn} A_{m,n+2}$$

$$- \gamma A_{mn} B_{m-2,n} + A_{mn} B_{m-2,n-2} - A_{mn} B_{m,n-2} + A_{mn} B_{n-2,n+2} - A_{mn} B_{m,n+2}\Big)\Big)$$

$$+\frac{t_c}{2} G_{cxz} \frac{\pi^2 b}{4a}\Big((m+1)^2\Big(\gamma A_{mn}^2 - A_{mn} A_{m,n-2} - \gamma A_{mn} A_{m+2,n} + A_{mn} A_{m+2,n-2} - A_{mn} A_{m,n+2} + A_{mn} A_{m+2,n+2}\Big)$$

$$+(m-1)^2\Big(\gamma A^2_{mn} - \gamma A_{mn} A_{m-2,n} + A_{mn} A_{m-2,n-2} - A_{mn} A_{m,n-2} + A_{mn} A_{m-2,n+2} - A_{mn} A_{m,n+2}\Big)\Big)$$

$$+\frac{t_c}{2} G_{cyz} \frac{ab}{4}\Big((n+1)^2\Big(\theta C_{mn}^2 - \theta C_{mn} C_{m,n+2} - C_{mn} C_{m-2,n} + C_{mn} C_{m-2,n+2} - C_{mn} C_{m+2,n} + C_{mn} C_{m+2,n+2}\Big)$$

$$+(n-1)^2\Big(\theta C_{mn}^2 - \theta C_{mn} C_{m,n-2} + C_{mn} C_{m-2,n-2} - C_{mn} C_{m-2,n} + C_{mn} C_{m+2,n-2} - C_{mn} C_{m+2,n}\Big)\Big)$$

$$+ t_c G_{cyz} \frac{\pi a}{8}\Big(n+1)^2\Big(2\theta C_{mn} A_{mn} - \theta C_{mn} A_{m,n+2} - C_{mn} A_{m-2,n} + C_{mn} A_{m-2,n+2} - C_{mn} A_{m+2,n} + C_{mn} A_{m+2,n+2}$$

$$- \theta A_{mn} C_{m,n+2} - A_{mn} C_{m-2,n} + A_{mn} C_{m-2,n+2} - A_{mn} C_{m+2,n} + A_{mn} C_{m+2,n+2}\Big)$$

$$+(n-1)^2\Big(\theta C_{mn} A_{mn} - \theta C_{mn} A_{m,n-2} + C_{mn} A_{m-2,n-2} - C_{mn} A_{m-2,n} + C_{mn} A_{m+2,n-2} - C_{mn} A_{m+2,n}$$

$$- \theta A_{mn} C_{m,n-2} + A_{mn} C_{m-2,n-2} - A_{mn} C_{m-2,n} + A_{mn} C_{m+2,n-2} - A_{mn} C_{m+2,n}\Big)\Big)$$

$$+ \frac{t_c}{2} G_{cyz} \frac{\pi^2 a}{4b}\Big((n+1)^2\Big(\theta A_{mn}^2 - \theta A_{mn} A_{m,n+2} - A_{mn} A_{m-2,n} + A_{mn} A_{m-2,n+2} + A_{mn} A_{m+2,n+2} - A_{mn} A_{m+2,n}\Big)$$

$$+(n-1)^2\Big(\theta A_{mn}^2 - \theta A_{mn} A_{m,n-2} + A_{mn} A_{m-2,n-2} - A_{mn} A_{m-2,n} + A_{mn} A_{m+2,n-2} - A_{mn} A_{m+2,n}\Big)\Big)$$

$$+\frac{1}{2}(N_x)\ \frac{\pi^2 b}{4a}\Big((m+1)^2\big(\gamma A^2_{mn} - A_{mn}A_{m,n-2} - \gamma A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m,n+2} + A_{mn}A_{m+2,n+2}\big)$$

$$+(m-1)^2\big(\gamma A^2_{mn} - \gamma A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m,n+2}\big)\Big)$$

$$+\frac{1}{2}(N_y)\ \frac{\pi^2 a}{4b}\Big((n+1)^2\big(\theta A^2_{mn} - \theta A_{mn}A_{m,n+2} - A_{mn}A_{m-2,n} + A_{mn}A_{m-2,n+2} - A_{mn}A_{m+2,n} + A_{mn}A_{m+2,n+2}\big)$$

$$+(n-1)^2\big(\theta A^2_{mn} - \theta A_{mn}A_{m,n-2} + A_{mn}A_{m-2,n-2} - A_{mn}A_{m-2,n} + A_{mn}A_{m+2,n-2} - A_{mn}A_{m+2,n}\big)\Big)$$

$$+\Big\{\sum_s \Delta_{ns}\ N_{xB}\ \frac{b}{2a}\Big((m+1)^2\Big(A_{mn}A_{ms}\Big(\frac{1}{(n-s)^2}+\frac{1}{(n+s-2)^2}-\frac{1}{(n-s-2)^2}+\frac{1}{(n-s)^2}-\frac{1}{(n+s)^2}-\frac{1}{(n-s+2)^2}-\frac{1}{(n+s)^2}+\frac{1}{(n+s+2)^2}\Big)$$

$$+A_{mn}A_{m+2,s}\Big(\frac{1}{(n+s)^2}-\frac{1}{(n-s)^2}-\frac{1}{(n+s-2)^2}+\frac{1}{(n-s-2)^2}+\frac{1}{(n+s+2)^2}+\frac{1}{(n+s)^2}-\frac{1}{(n+s)^2}-\frac{1}{(n-s+2)^2}\Big)\Big)$$

$$+(m-1)^2\Big(A_{mn}A_{ms}\Big(\frac{1}{(n-s)^2}+\frac{1}{(n+s-2)^2}-\frac{1}{(n-s-2)^2}-\frac{1}{(n+s)^2}-\frac{1}{(n-s+2)^2}-\frac{1}{(n+s)^2}+\frac{1}{(n-s)^2}+\frac{1}{(n+s+2)^2}\Big)$$

$$+A_{mn}A_{m-s,s}\Big(\frac{1}{(n+s)^2}-\frac{1}{(n-s)^2}-\frac{1}{(n+s-2)^2}+\frac{1}{(n-s-2)^2}+\frac{1}{(n-s+2)^2}+\frac{1}{(n+s)^2}-\frac{1}{(n-s)^2}-\frac{1}{(n+s+2)^2}\Big)\Big)\Big)\Big)$$

$$+\sum_r \Delta_{mr}\ N_{yB}\ \frac{a}{2b}\Big((n+1)^2\Big(A_{mn}A_{rn}\Big(\frac{1}{(m-r)^2}+\frac{1}{(m+r-2)^2}-\frac{1}{(m-r-2)^2}-\frac{1}{(m+r)^2}-\frac{1}{(m-r+2)}-\frac{1}{(m+r)^2}$$

$$+\frac{1}{(m-r)^2}+\frac{1}{(m+r+2)^2}\Big)+A_{mn}A_{r,n+2}\Big(\frac{1}{(m+r)^2}-\frac{1}{(m-r)^2}-\frac{1}{(m+r-2)^2}+\frac{1}{(m-r-2)^2}$$

$$+\frac{1}{(m-r+2)^2}+\frac{1}{(m+r)^2}-\frac{1}{(m-r)^2}-\frac{1}{(m+r+2)^2}\Big)\Big)$$

$$+(n-1)^2\Big(A_{mn}A_{rn}\Big(\frac{1}{(m-r)^2}+\frac{1}{(m+r-2)^2}-\frac{1}{(m-r-2)^2}-\frac{1}{(m+r)^2}-\frac{1}{(m-r+2)}-\frac{1}{(m+r)^2}$$

$$+\frac{1}{(m-r)^2}+\frac{1}{(m+r+2)^2}\Big)+A_{mn}A_{r,n-2}\Big(\frac{1}{(m+r)^2}-\frac{1}{(m-r)^2}-\frac{1}{(m+r-2)^2}+\frac{1}{(m-r-2)^2}$$

$$+\frac{1}{(m-r+2)^2}+\frac{1}{(m+r)^2}-\frac{1}{(m-r)^2}-\frac{1}{(m+r+2)^2}\Big)\Big)\Big)$$

$$+\sum_r \sum_s \Delta_{mr} \Delta_{ns} N_{xy} \; 2 A_{mn} A_{rs} \Bigg(\frac{(m+1)^2 (s+1)^2 + (r-1)^2 (n-1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s+1)^2-(n-1)^2\right)} - \frac{(m+1)^2 (s-1)^2 + (r-1)^2 (n-1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s-1)^2-(n-1)^2\right)}$$

$$- \frac{(m+1)^2 (s+1)^2 + (r+1)^2 (n-1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s+1)^2-(n-1)^2\right)} + \frac{(m+1)^2 (s-1)^2 + (r+1)^2 (n-1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s-1)^2-(n-1)^2\right)} - \frac{(m+1)^2 (s+1)^2 + (r-1)^2 (n+1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s+1)^2-(n+1)^2\right)}$$

$$+ \frac{(m+1)^2 (s-1)^2 + (r-1)^2 (n+1)^2}{\left((m+1)^2-(r-1)^2\right)\left((s-1)^2-(n-1)^2\right)} + \frac{(m+1)^2 (s+1)^2 + (r+1)^2 (n+1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s+1)^2-(n+1)^2\right)} - \frac{(m+1)^2 (s-1)^2 + (r+1)^2 (n+1)^2}{\left((m+1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)}$$

$$- \frac{(m-1)^2 (s+1)^2 + (r-1)^2 (n-1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s+1)^2-(n-1)^2\right)} + \frac{(m-1)^2 (s-1)^2 + (r-1)^2 (n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)} + \frac{(m-1)^2 (s+1)^2 + (r+1)^2 (n-1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s+1)^2-(n-1)^2\right)}$$

$$- \frac{(m-1)^2 (s-1)^2 + (r+1)^2 (n-1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n-1)^2\right)} + \frac{(m-1)^2 (s+1)^2 + (r-1)^2 (n+1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s-1)^2-(n+1)^2\right)} - \frac{(m-1)^2 (s-1)^2 + (r-1)^2 (n+1)^2}{\left((m-1)^2-(r-1)^2\right)\left((s-1)^2-(n+1)^2\right)}$$

$$- \frac{(m-1)^2 (s+1)^2 + (r+1)^2 (n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s+1)^2-(n+1)^2\right)} + \frac{(m-1)^2 (s-1)^2 + (r+1)^2 (n+1)^2}{\left((m-1)^2-(r+1)^2\right)\left((s-1)^2-(n+1)^2\right)} \Bigg) \Bigg| \Bigg]$$

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